Jacobians and Hessians of Mean Value Coordinates for Closed Triangular Meshes

Additional Material

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Abstract

Mean Value Coordinates provide an efficient mechanism for the interpolation of scalar functions defined on orientable domains with non-convex boundary. They present several interesting features, including the simplicity and speed that yield from their closed-form expression. In several applications though, it is desirable to enforce additional constraints involving the partial derivatives of the interpolated function, as done in the case of the Green Coordinates approximation scheme [Ben-Chen et al.(2009)Ben-Chen, Weber, and Gotsman] for interactive 3D model deformation.

In this paper, we introduce the analytic expressions of the Jacobian and the Hessian of functions interpolated through Mean Value Coordinates. We provide these expressions both for the 2D and 3D case. We also provide a thorough analysis of their degenerate configurations along with accurate approximations of the partial derivatives in these configurations. Extensive numerical experiments show the accuracy of our derivation. In particular, we illustrate the improvements of our formulae over a variety of Finite Difference schemes in terms of precision and usability. We demonstrate the utility of this derivation in several applications, including cage-based implicit 3D model deformations (i.e. *Variational MVC deformations*). This technique allows for easy and interactive model deformations with sparse positional, rotational and smoothness constraints. Moreover, the cages produced by the algorithm can be directly re-used for further manipulations, which makes our framework directly compatible with existing software supporting Mean Value Coordinates based deformations.

Disclaimer

This material contains the full derivation of the derivatives of Mean-Value Coordinates for piecewise linear meshes, in 2D (where the mesh is a closed polygon) and in 3D (where the mesh is a closed triangle mesh). No application is discussed in this document.

The derivation is relatively involved, and due to a lack of space, only the final results appear in the corresponding article.

Overview

We first review Mean Value Coordinates in Sec. 1. The core contribution of our work, the derivation of the Jacobian and Hessian, is presented in Sec. 2 through 5. In particular, Sec. 3 and 4 provide the specific results for the 2D and 3D cases respectively.

As the derivation of the Jacobian and the Hessian is relatively involved, for the reader's convenience, we highlighted the final expressions with rectangular boxes, whereas the final expressions for degenerate cases are highlighted with a ellipsoidal box.

All the mathematical details of the derivation that were left out for readability are given in Sec. 6, 7, 8 and 9. We also present empirical evaluation of the derivatives in 3D by comparing our results with Finite Differences in Sec. 10.

1 Background

In this section, we review the formulation of Mean Value Coordinates in 2D and 3D [Ju et al.(2005)Ju, Schaefer, and Warren].

1.1 Mean Value Coordinates

Similar to [Ju et al.(2005)Ju, Schaefer, and Warren], we note $p[x]$ a parameterization of a closed $(d-1)$-manifold mesh (the cage) M embedded in $\mathbb{R}^d$, where x is a $(d-1)$-dimensional parameter, and n_x the unit outward normal at x. Let η be a point in $\mathbb{R}^d$ expressed as a linear combination of the positions p_i of the vertices of the cage M:

$$\eta = \frac{\sum_i w_i p_i}{\sum_i w_i} = \sum_i \lambda_i p_i$$

where λ_i is the *barycentric coordinate* of η with respect to the vertex i.

Let $\phi_i[x]$ be the linear function on M which maps the vertex i to 1 and all other vertices to 0.

The definition of the coordinates λ_i should guarantee *linear precision* (i.e. $\eta = \sum_i \lambda_i p_i$).

Similar to [Ju et al.(2005)Ju, Schaefer, and Warren], we note $B_\eta(M)$ the projection of the manifold M onto the unit sphere centered around η, and $dS_\eta(x)$ the infinitesimal element of surface on this sphere at the projected point ($dS_\eta(x) = \frac{(p[x]-\eta)^t \cdot n_x}{|p[x]-\eta|^3} dx$ in 3D)

Since $\int_{B_\eta(M)} \frac{p[x]-\eta}{|p[x]-\eta|} dS_\eta(x) = 0$ (the integral of the unit outward normal onto the unit sphere is 0 in any dimension $d \geq 2$), the following equation holds:

$$\eta = \frac{\int_{B_\eta(M)} \frac{p[x]}{|p[x]-\eta|} dS_\eta(x)}{\int_{B_\eta(M)} \frac{1}{|p[x]-\eta|} dS_\eta(x)}$$

By writing $p[x] = \sum_i \phi_i[x] p_i \ \forall x$, with $\sum_i \phi_i[x] = 1$, we have:

$$\eta = \frac{\sum_i \int_{B_\eta(M)} \frac{\phi_i[x]}{|p[x]-\eta|} dS_\eta(x) p_i}{\int_{B_\eta(M)} \frac{1}{|p[x]-\eta|} dS_\eta(x)}$$

The coordinates λ_i are then given by:

$$\lambda_i = \frac{\int_{B_\eta(M)} \frac{\phi_i[x]}{|p[x]-\eta|} dS_\eta(x)}{\int_{B_\eta(M)} \frac{1}{|p[x]-\eta|} dS_\eta(x)}$$

and the weights w_i such that $\lambda_i = \frac{w_i}{\sum_i w_i}$ are given by:

$$w_i = \int_{B_\eta(M)} \frac{\phi_i[x]}{|p[x]-\eta|} dS_\eta(x) \tag{1}$$

This definition guarantees linear precision [Ju et al.(2005)Ju, Schaefer, and Warren]. It also provides a linear interpolation of the function prescribed at the vertices of the cage onto its simplices and it smoothly extends it to the entire space. This construction of Mean Value Coordinates is valid in any dimension $d \geq 2$. In the following, we present their computation in 2D and in 3D, as they were introduced in [Ju et al.(2005)Ju, Schaefer, and Warren].

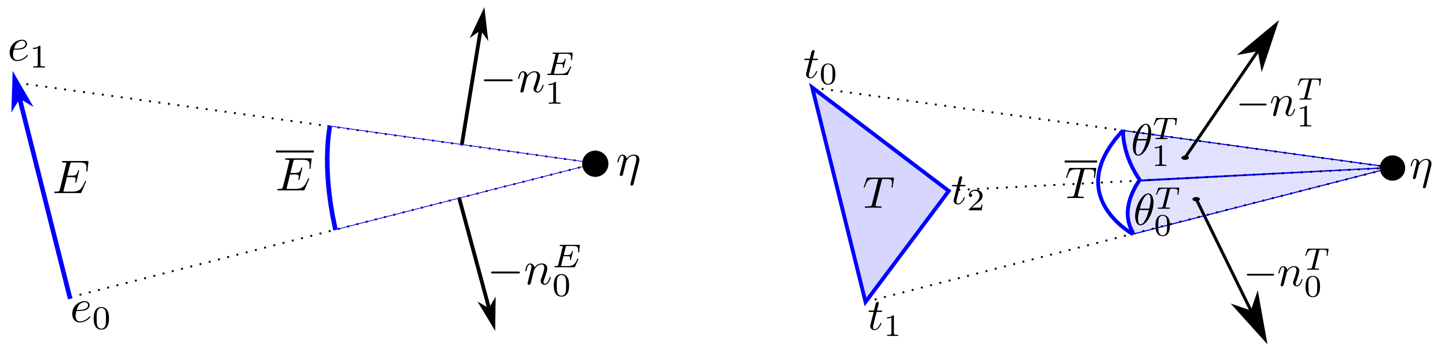


Figure 1: Spherical edge Ē (left) and triangle T̄(right).

1.2 3D Mean Value Coordinates computation

The support of the function $\phi_i[x]$ is only composed of the adjacent triangles to the vertex i (noted $N1(i)$). Eq. 1 can be re-written as $w_i = \sum_{T \in N1(i)} w_i^T$, with

$$w_i^T = \int_{B_\eta(T)} \frac{\phi_i[x]}{|p[x] - \eta|} dS_\eta(x) \tag{2}$$

Given a triangle T with vertices t_1, t_2, t_3, the following equation holds:

$$\begin{aligned} \sum_j w_{t_j}^T (p_{t_j} - \eta) &= \int_{B_\eta(T)} \frac{\sum_j \phi_{t_j}[x](p_{t_j} - \eta)}{|p[x] - \eta|} dS_\eta(x) \\ &= \int_{B_\eta(T)} \frac{p[x] - \eta}{|p[x] - \eta|} dS_\eta(x) \triangleq m^T \end{aligned} \tag{3}$$

The latter integral is the integral of the unit outward normal on the spherical triangle $\overline{T} = B_\eta(T)$ (see Fig. 1). By noting the unit normal as $n_i^T = \frac{N_i^T}{|N_i^T|}$, with $N_i^T \triangleq (p_{t_{i+1}} - \eta) \wedge (p_{t_{i+2}} - \eta)$ (see Fig.1), m^T is given by (since the integral of the unit normal on a closed surface is always 0):

$$m^T = \sum_i \frac{1}{2} \theta_i^T n_i^T \tag{4}$$

As suggested in [Ju et al.(2005)Ju, Schaefer, and Warren], the weights $w_{t_j}^T$ can be obtained by noting A^T the 3x3 matrix $\{p_{t_1} - \eta, p_{t_2} - \eta, p_{t_3} - \eta\}$ (where ^{t} denotes the transpose):

$$\{w_{t_1}^T, w_{t_2}^T, w_{t_3}^T\}^t = {A^T}^{-1} \cdot m^T$$

Since ${N_i^T}^t \cdot (p_{t_j} - \eta) = 0 \;\; \forall i \neq j$ and ${N_i^T}^t \cdot (p_{t_i} - \eta) = \det(A^T) \;\; \forall i$, the final expression for the weights is given by:

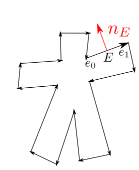
$$w_{t_i}^T = \frac{{N_i^T}^t \cdot m^T}{{N_i^T}^t \cdot (p_{t_i} - \eta)} = \frac{{N_i^T}^t \cdot m^T}{\det(A^T)} \quad \forall \eta \notin Support(T)$$

where $Support(T)$ denotes the support plane of T, i. e. $Support(T) = \{\eta \in \mathbb{R}^3 | \det(A^T)(\eta) = 0\}$.

1.3 2D Mean Value Coordinates computation

Let I_2 be the 2×2 identity matrix and $R_{\frac{\pi}{2}}$ the rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

In 2D, the orientation of an edge $E = e_0 e_1$ of a closed polygon is defined by the normal n_E:



$$n_E = \frac{R_{\frac{\pi}{2}}(p_{e_1} - p_{e_0})}{|p_{e_1} - p_{e_0}|}$$

It defines consistently the *interior* and the *exterior* of the closed polygon. Then, similarly to the 3D case:

$$\sum_j w_{e_j}^E \cdot (p_{e_j} - \eta) = m^E = \sum_j n_j^E \tag{5}$$

with:
$$n_j^E = \frac{N_j^E}{|N_j^E|},\ N_0^E = R_{\frac{\pi}{2}}(\eta - p_{e_0}),\ N_1^E = -R_{\frac{\pi}{2}}(\eta - p_{e_1})$$

Therefore:
$$m^E = R_{\frac{\pi}{2}}\left(\frac{\eta - p_{e_0}}{|\eta - p_{e_0}|} - \frac{\eta - p_{e_1}}{|\eta - p_{e_1}|}\right) \tag{6}$$

Since $(p_{e_j} - \eta)^t \cdot N_j^E = 0$ (Fig.1), we obtain w_i^E with:

$$w_{e_i}^E = \frac{{m^E}^t \cdot N_{i+1}^E}{(p_{e_i} - \eta)^t \cdot N_{i+1}^E} \tag{7}$$

2 Derivation Overview

In the following, we present the main contribution of the paper: the derivation of the Jacobians and the Hessians of Mean Value Coordinates. In this section, we briefly give an overview of the derivation.

Let $f : M \to \mathbb{R}^d$ be a piecewise linear field defined on M (in 2D, M is a closed polygon, in 3D, M is a closed triangular mesh). As reviewed in the previous section, f can be smoothly interpolated with Mean Value Coordinates for any point η of the Euclidean space:
$$f(\eta) = \sum_i \lambda_i \cdot f(p_i)$$

Then, the Jacobian and the Hessian of f, respectively noted Jf and Hf, are expressed as the linear tensor product of the values $f(p_i)$ with the gradient $\vec{\nabla}\lambda_i$ and the Hessian $H\lambda_i$ of the coordinates respectively:

$$\begin{cases} Jf &= \sum_i f(p_i) \cdot \vec{\nabla}\lambda_i{}^t \\ Hf &= \sum_i f(p_i) \cdot H\lambda_i \end{cases}$$

Since $\lambda_i = \frac{w_i}{\sum_j w_j}$,

$$\vec{\nabla}\lambda_i = \frac{\vec{\nabla}w_i}{\sum_j w_j} - \frac{w_i \cdot \sum_j \vec{\nabla}w_j}{(\sum_j w_j)^2} \tag{8}$$

Then, the Hessian can be obtained with the following equations

$$\begin{aligned} H\lambda_i =& \frac{Hw_i}{\sum_j w_j} - \frac{w_i \sum_j Hw_j}{(\sum_j w_j)^2} \\ &- \frac{\vec{\nabla}w_i \cdot \sum_j \vec{\nabla}(w_j)^t + \sum_j \vec{\nabla}(w_j) \cdot \vec{\nabla}w_i{}^t}{(\sum_j w_j)^2} \\ &+ \frac{2w_i(\sum_j \vec{\nabla}w_j) \cdot (\sum_j \vec{\nabla}w_j)^t}{(\sum_j w_j)^3} \end{aligned} \tag{9}$$

The above expressions are general and valid for the 2D and 3D cases. Thus, in order to derive a closed-form expression of the gradient and the Hessian of the Mean Value Coordinates λ_i, one needs to derive the expressions of $\vec{\nabla}w_i$ (Eq. 8) and Hw_i (Eq. 9). The expressions of these terms are derived in Sec. 3.1 and Sec. 3.3 respectively for the 2D case and in Sec. 4.1 and Sec. 4.3 for the 3D case.

Properties

Functions interpolated by means of Mean Value Coordinates as previously described have the following properties:

1. they are interpolant on M
2. they are defined everywhere in $\mathbb{R}^n$
3. they are C^∞ everywhere not on M
4. they are C^0 on the edges (resp. vertices) of M in 3D (resp. in 2D)

Since these are interpolant of piecewise linear functions defined on a piecewise linear domain, they cannot be differentiable on the edges of the triangles (resp. the vertices of the edges) of the cage in 3D (resp. in 2D). Although, as they are continuous everywhere, they may admit in these cases **directional derivatives** like for almost all continuous functions, but as they are of no use at all in general, we won't present in this paper these quantities. Recall that the directional derivative is the value $\partial f_u(\eta) = \lim_{\epsilon \to 0^+} \frac{f(\eta + \epsilon \cdot u) - f(\eta)}{\epsilon}$, with $u \in \mathbb{R}^3, ||u|| = 1, \epsilon \in \mathbb{R}$, **which strongly depends on the orientation of the vector** u **where the limit is considered**. These derivatives cannot be used to evaluate the neighborhood of a the function around the point in general with a single gradient (or Jacobian if the function is multi-dimensional).

In this paper, we provide formulae for the 1^{st} and 2^{nd} order derivatives of the Mean Value Coordinates everywhere in space but on the cage.

3 MV-Gradients and Hessians in 2D

In the following, we note (pq) the line going through the points p and q, and $[pq]$ the line segment between them.

3.1 Expression of the MV-gradients

By differentiating Eq. 5, we obtain:

$$\sum_j (p_{e_j} - \eta) \cdot \overrightarrow{\bigtriangledown} {w_{e_j}^E}^t = Jm^E + \sum_j w_{e_j}^E I_2 = B^E(\eta) \tag{10}$$

Jm^E is given by differentiating Eq. 6:

$$\begin{aligned} Jm^E = R_{\frac{\pi}{2}}(& \frac{I_2}{|\eta - p_{e_0}|} - \frac{I_2}{|\eta - p_{e_1}|} \\ & - \frac{(\eta - p_{e_0}) \cdot (\eta - p_{e_0})^t}{|\eta - p_{e_0}|^3} \\ & + \frac{(\eta - p_{e_1}) \cdot (\eta - p_{e_1})^t}{|\eta - p_{e_1}|^3}) \end{aligned} \tag{11}$$

Then, in the general case where $(p_{e_i} - \eta)^t \cdot N_{i+1}^E \neq 0$, the gradient of the weights is given by the following expression:

$$\boxed{\overrightarrow{\bigtriangledown} w_{e_i}^E = \frac{{B^E}^t \cdot N_{i+1}^E}{(p_{e_i} - \eta)^t \cdot N_{i+1}^E} \quad \forall \eta \notin (p_{e_0} p_{e_1})} \tag{12}$$

3.2 Special case: $\eta \in (p_{e_0} p_{e_1}), \notin [p_{e_0} p_{e_1}]$

The special case where $(p_{e_i} - \eta)^t \cdot N_{i+1}^E = 0$ only occurs when η lies on the same line as the edge E (η lies on the *support* of the edge E, noted $Support(E) = (p_{e_0} p_{e_1})$). As discussed in Sec. 2, we omit the case where $\eta \in [e_0 e_1]$. Since the length of $\bar{E}$ is zero (see Fig. 1), for all $\eta \in (p_{e_0} p_{e_1}), \notin [p_{e_0} p_{e_1}]$, $w_{e_i}^E(\eta) = 0 \; \forall i = 0, 1$.

Similarly, $\overrightarrow{\bigtriangledown} w_{e_i}^E$ and n_E are collinear, then:

$$\overrightarrow{\bigtriangledown} w_{e_i}^E = \frac{\partial (w_{e_i}^E(\eta + \epsilon n_E))}{\partial \epsilon}_{|\epsilon \to 0} n_E$$

A closed-form expression can be derived from the above equation with Taylor expansions. For conciseness, the details of this derivation are given in section 9 and only the final expression is given here:

$$\boxed{\overrightarrow{\bigtriangledown} w_{e_i}^E = (\sum_j \frac{{N_j^E}^t \cdot N_{i+1}^E}{2|E||N_j^E|^3} + \frac{(-1)^{i+j}}{|E||N_j^E|}) n_E \quad \forall \eta \in (p_{e_0} p_{e_1}), \notin [p_{e_0} p_{e_1}]} \tag{13}$$

3.3 Expression of the MV-Hessians

By differentiating Eq. 10 successively with regards to $c = \{x, y\}$, we obtain:

$$\begin{aligned} & \sum_i \partial_c (p_{e_i} - \eta) \cdot \overrightarrow{\bigtriangledown} {w_{e_i}^E}^t + \sum_i (p_{e_i} - \eta) \cdot \partial_c (\overrightarrow{\bigtriangledown} {w_{e_i}^E}^t) \\ & = \partial_c (Jm^E) + \sum_i \partial_c (w_{e_i}^E) \cdot I_2 \end{aligned}$$

$$\begin{cases} \sum_i (p_{e_i} - \eta) \cdot \partial_x (\overrightarrow{\bigtriangledown} {w_{e_i}^E}^t) & = C_x^E \\ \sum_i (p_{e_i} - \eta) \cdot \partial_y (\overrightarrow{\bigtriangledown} {w_{e_i}^E}^t) & = C_y^E \end{cases} \tag{14}$$

with

$$\begin{cases} C_x^E & = \sum_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \overrightarrow{\bigtriangledown} {w_{e_i}^E}^t + \partial_x (Jm^E) + \sum_i \partial_x (w_{e_i}^E) \cdot I_2 \\ C_y^E & = \sum_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \overrightarrow{\bigtriangledown} {w_{e_i}^E}^t + \partial_y (Jm^E) + \sum_i \partial_y (w_{e_i}^E) \cdot I_2 \end{cases}$$

Expressions have been provided for all of the terms appearing in the above equations, except for the derivatives of Jm^E.

Expression of $\partial_c(Jm^E)$

By differentiating Eq. 11, we obtain:

$$\begin{aligned}\partial_c(Jm^E) = & R_{\frac{\pi}{2}} \cdot (\frac{(\eta - p_{e_1})_{(c)} I_2}{|\eta - p_{e_1}|^3} - \frac{(\eta - p_{e_0})_{(c)} I_2}{|\eta - p_{e_0}|^3} \\ & - \frac{\delta_c \cdot (\eta - p_{e_0})^t + (\eta - p_{e_0}) \cdot \delta_c{}^t}{|\eta - p_{e_0}|^3} \\ & + \frac{3(\eta - p_{e_0})_{(c)} (\eta - p_{e_0}) \cdot (\eta - p_{e_0})^t}{|\eta - p_{e_0}|^5} \\ & + \frac{\delta_c \cdot (\eta - p_{e_1})^t + (\eta - p_{e_1}) \cdot \delta_c{}^t}{|\eta - p_{e_1}|^3} \\ & - \frac{3(\eta - p_{e_1})_{(c)} (\eta - p_{e_1}) \cdot (\eta - p_{e_1})^t}{|\eta - p_{e_1}|^5})\end{aligned}$$

where $\delta_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\delta_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $X_{(c)}$ denotes the c^{th} coordinate of the vector X.

3.4 Final expression of $Hw_{e_i}^E$

From Eq. 14, $((p_{e_i} - \eta)^t \cdot N_i^E = 0$, see Fig. 1) we obtain:

$$\left\{ \begin{array}{ll} \partial_x(\overrightarrow{\nabla} w_{e_i}^E) & = \frac{C_x^{E\,t} \cdot N_{i+1}^E}{(p_{e_i} - \eta)^t \cdot N_{i+1}^E} \\ \partial_y(\overrightarrow{\nabla} w_{e_i}^E) & = \frac{C_y^{E\,t} \cdot N_{i+1}^E}{(p_{e_i} - \eta)^t \cdot N_{i+1}^E} \end{array} \right\} \quad \forall \eta \notin (p_{e_0} p_{e_1}).$$

Finally:

$$\boxed{Hw_{e_i}^E = \begin{pmatrix} \partial_x(\overrightarrow{\nabla} w_{e_i}^E)^t \\ \partial_y(\overrightarrow{\nabla} w_{e_i}^E)^t \end{pmatrix} \quad \forall \eta \notin (p_{e_0} p_{e_1})} \quad (15)$$

3.5 Special case: $\eta \in (p_{e_0} p_{e_1}), \notin [p_{e_0} p_{e_1}]$

When η lies on the support of the edge E, $\overrightarrow{\nabla} w_{e_i}^E(\eta) = dw_{e_i}^E n_E$, where $dw_{e_i}^E = \frac{\partial(w_{e_i}^E(\eta + \epsilon n_E))}{\partial \epsilon}_{|\epsilon \to 0}$ is a scalar term obtained in the special case of Sec. 3.2 (see section 9 for the derivation of $dw_{e_i}^E$).

To compute the value of the Hessian, **it is however not enough to write $n_E \cdot \overrightarrow{\nabla}(dw_{e_i}^E)^t$, and this last value is not a symmetric matrix**. By deriving the expressions of the gradient and the Hessian (writing it by replacing the integration on the unit sphere by the integration on the manifold, using $dS_\eta(x) = \frac{(p[x]-\eta)^t \cdot n_x}{|p[x]-\eta|^2} dx$ in 2D, see Sec. 1), we can see that

$$w_{e_i}^E = \int_{x \in E} \frac{\phi_i[x](p[x] - \eta)^t \cdot n_x}{|p[x] - \eta|^3} dx$$

$$\overrightarrow{\nabla} w_{e_i}^E = - \int_{x \in E} \frac{\phi_i[x] n_x}{|p[x] - \eta|^3} dx$$
$$-3 \int_{x \in E} \frac{\phi_i[x](p[x] - \eta)^t \cdot n_x}{|p[x] - \eta|^5} (\eta - p[x]) dx$$

The second integral is 0 in the particular case of $\eta \in (p_{e_0} p_{e_1}), \notin [p_{e_0} p_{e_1}]$, since $(p[x] - \eta)^t \cdot n_x \forall x \in E$, thus we can identity the first integral to be $dw_i^E n_E$, dw_i^E being the function computed in Sec. 3.2.

If we differentiate the gradient to obtain the value of the Hessian, we can see that

$$
\begin{aligned}
Hw^E_{e_i} = 3\int_{x\in E}\frac{\phi_i[x]n_x\cdot(\eta-p[x])^t}{|p[x]-\eta|^5}dx\\
-3\int_{x\in E}\frac{\phi_i[x](p[x]-\eta)^t\cdot n_x}{|p[x]-\eta|^5}dxI_2\\
+3\int_{x\in E}\frac{\phi_i[x](\eta-p[x])\cdot n_x^t}{|p[x]-\eta|^5}dx\\
+15\int_{x\in E}\frac{\phi_i[x](p[x]-\eta)^t\cdot n_x*(p[x]-\eta)\cdot(p[x]-\eta)^t}{|p[x]-\eta|^5}dx
\end{aligned}
$$

Once again, we make the remark that the second and the fourth integrals are 0 in the special case of $\eta\in(p_{e_0}p_{e_1}),\notin[p_{e_0}p_{e_1}]$, and we can identity the first integral as $n_E\cdot\overrightarrow{\triangledown}(dw^E_{e_i})^t$ in that particular case only (and the third integral is the transpose of the first).

Finally:

$$
\boxed{\begin{aligned}
Hw^E_{e_i} = n_E\cdot\overrightarrow{\triangledown}(dw^E_{e_i})^t+\overrightarrow{\triangledown}(dw^E_{e_i})\cdot n_E^t\\
\forall\eta\in(p_{e_0}p_{e_1}),\notin[p_{e_0}p_{e_1}]
\end{aligned}}
$$

where

$$
\begin{aligned}
\overrightarrow{\triangledown}(dw^E_{e_i}) =&\sum_j\frac{JN^E_{i+1}{}^t\cdot N^E_j+JN^{E^t}_j\cdot N^E_{i+1}}{2|E||N^E_j|^3}\\
&-\sum_j\frac{3(N^{E^t}_j\cdot N^E_{i+1})JN^{E^t}_j\cdot N^E_j}{2|E||N^E_j|^5}\\
&+\sum_j\frac{(-1)^{i+j+1}JN_jj^{E^t}\cdot N^E_j}{|E||N^E_j|^3}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\triangledown}(dw^E_{e_i}) =&R_{\frac{\pi}{2}}\cdot\sum_j\frac{3(-1)^{i+1}N^E_j+(-1)^jN^E_{i+1}}{2|E||N^E_j|^3}\\
&+R_{\frac{\pi}{2}}\cdot\sum_j\frac{3(-1)^{j+1}(N^{E^t}_j\cdot N^E_{i+1})N^E_j}{2|E||N^E_j|^5}
\end{aligned}
$$

4 MV-Gradients and Hessians in 3D

4.1 Expression of the MV-Gradients

Instead of differentiating Eq. 2 (which involves the computation of an integral), we derive the Jacobian from Eq. 3 (by noting $B^T \triangleq Jm^T + \sum_j w^T_{t_j} \cdot I_3$):

$$\sum_j (p_{t_j} - \eta) \cdot \vec{\nabla} w^T_{t_j}{}^t = B^T \tag{16}$$

From the above expression, we obtain: (again, $N^T_i{}^t \cdot (p_{t_j} - \eta) = 0 \ \forall j \neq i$ and $N^T_i{}^t \cdot (p_{t_i} - \eta) = \det(A^T) \ \forall i$, see Fig. 1)

$$\boxed{\vec{\nabla} w^T_{t_i} = \frac{B^T{}^t \cdot N^T_i}{\det(A^T)} \quad (17) \qquad \forall \eta \notin Support(T)}$$

At this point, the Jacobian of the vector m^T is required for the expression of B^T.

Expression of Jm^T

From Eq. 4, we have:

$$m^T = \sum_j \frac{\theta^T_j n^T_j}{2} = \sum_j \frac{\theta^T_j N^T_j}{2|N^T_j|}$$

Then, we obtain:

$$\begin{aligned} Jm^T = & \sum_j \frac{N^T_j \cdot \vec{\nabla} \theta^T_j{}^t}{2|N^T_j|} + \sum_j \frac{\theta^T_j JN^T_j}{2|N^T_j|} \\ & - \sum_j \frac{\theta^T_j N^T_j \cdot (JN^T_j{}^t \cdot N^T_j)^t}{2|N^T_j|^3} \end{aligned} \tag{18}$$

Derivatives of N^T_j

From the expression of N^T_j (Sec. 1.2), we obtain:

$$\begin{aligned} N^T_j(\eta + d\eta) = & (p_{t_{j+1}} - \eta - d\eta) \wedge (p_{t_{j+2}} - \eta - d\eta) \\ = & (p_{t_{j+1}} - \eta) \wedge (p_{t_{j+2}} - \eta) + (p_{t_{j+2}} - p_{t_{j+1}}) \wedge d\eta \\ = & N^T_j(\eta) + (p_{t_{j+2}} - p_{t_{j+1}}) \wedge d\eta. \end{aligned}$$

Therefore

$$JN^T_j = (p_{t_{j+2}} - p_{t_{j+1}})_{[\wedge]} \tag{19}$$

where $k_{[\wedge]}$ is the skew 3×3 matrix (i.e. $k_{[\wedge]}{}^t = -k_{[\wedge]}$) such that $k_{[\wedge]} \cdot u = k \wedge u \ \ \forall k, u \in \mathbb{R}^3$.
In particular, we see from Eq. 19 that N^T_j ***admits a ull second order derivative***.

Expression of $\vec{\nabla} \theta^T_j$

The term $\vec{\nabla} \theta^T_j$ can be derived from the following expressions:

$$\begin{aligned} \sin(\theta^T_j) = S^T_j \quad , \quad S^T_j & \triangleq \frac{|(p_{t_{j+2}} - \eta) \wedge (p_{t_{j+1}} - \eta)|}{|p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|} = \frac{|N^T_j|}{|p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|} \\ \cos(\theta^T_j) = C^T_j \quad , \quad C^T_j & \triangleq \frac{(p_{t_{j+2}} - \eta)^t \cdot (p_{t_{j+1}} - \eta)}{|p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|} \end{aligned}$$

$$\cos(\theta^T_j) \cdot \vec{\nabla} \theta^T_j = \vec{\nabla} S^T_j \tag{20}$$

$$-\sin(\theta^T_j) \cdot \vec{\nabla} \theta^T_j = \vec{\nabla} C^T_j \tag{21}$$

$\overrightarrow{\bigtriangledown}\theta_j^T$ can be evaluated with Eq. 20 when $\theta_j^T \neq \pi/2$, and with Eq. 21 when $\theta_j^T \neq 0, \pi$. $\overrightarrow{\bigtriangledown}S_j^T$ and $\overrightarrow{\bigtriangledown}C_j^T$ are given by the following expressions:

$$
\begin{aligned}
\overrightarrow{\bigtriangledown}S_j^T = & \frac{J{N_j^T}^t \cdot N_j^T}{|N_j^T| \cdot |p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|} \\
& - \frac{|N_j^T| \cdot (\eta - p_{t_{j+2}})}{|p_{t_{j+2}} - \eta|^3 \cdot |p_{t_{j+1}} - \eta|} \\
& - \frac{|N_j^T| \cdot (\eta - p_{t_{j+1}})}{|p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|^3}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\bigtriangledown}S_j^T = & \frac{J{N_j^T}^t \cdot N_j^T}{|N_j^T| \cdot |p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|} \\
& - \frac{(\eta - p_{t_{j+2}}) \cdot \sin(\theta_j^T)}{|p_{t_{j+2}} - \eta|^2} \\
& - \frac{(\eta - p_{t_{j+1}}) \cdot \sin(\theta_j^T)}{|p_{t_{j+1}} - \eta|^2}
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\overrightarrow{\bigtriangledown}C_j^T = & \frac{2\eta - p_{t_{j+1}} - p_{t_{j+2}}}{|p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|} \\
& - \frac{(\eta - p_{t_{j+2}}) \cdot (p_{t_{j+2}} - \eta)^t \cdot (p_{t_{j+1}} - \eta)}{|p_{t_{j+2}} - \eta|^3 \cdot |p_{t_{j+1}} - \eta|} \\
& - \frac{(\eta - p_{t_{j+1}}) \cdot (p_{t_{j+2}} - \eta)^t \cdot (p_{t_{j+1}} - \eta)}{|p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|^3}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\bigtriangledown}C_j^T = & \frac{2\eta - p_{t_{j+1}} - p_{t_{j+2}}}{|p_{t_{j+2}} - \eta| \cdot |p_{t_{j+1}} - \eta|} \\
& - \frac{(\eta - p_{t_{j+2}}) \cdot \cos(\theta_j^T)}{|p_{t_{j+2}} - \eta|^2} \\
& - \frac{(\eta - p_{t_{j+1}}) \cdot \cos(\theta_j^T)}{|p_{t_{j+1}} - \eta|^2}
\end{aligned} \tag{23}
$$

From Eq. 20 and 21, we obtain:

$$\cos(\theta_j^T)\overrightarrow{\bigtriangledown}S_j^T - \sin(\theta_j^T)\overrightarrow{\bigtriangledown}C_j^T = \cos(\theta_j^T)^2\overrightarrow{\bigtriangledown}\theta_j^T + \sin(\theta_j^T)^2\overrightarrow{\bigtriangledown}\theta_j^T = \overrightarrow{\bigtriangledown}\theta_j^T$$

And by replacing the expressions of $\overrightarrow{\bigtriangledown}S_j^T$ and $\overrightarrow{\bigtriangledown}C_j^T$ on the left side of the equality by those of Eq. 22 and 23:

$$
\begin{aligned}
\overrightarrow{\bigtriangledown}\theta_j^T &= \frac{\cos(\theta_j^T)J{N_j^T}^t \cdot N_j^T}{|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta||N_j^T|} - \frac{\sin(\theta_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|} \\
&= \frac{\cos(\theta_j^T)\sin(\theta_j^T)J{N_j^T}^t \cdot N_j^T}{|N_j^T|^2} - \frac{\sin(\theta_j^T)^2(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{|N_j^T|}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{\bigtriangledown}\theta_j^T = & \frac{\cos(\theta_j^T)\sin(\theta_j^T)J{N_j^T}^t \cdot N_j^T}{|N_j^T|^2} \\
& - \frac{\sin(\theta_j^T)^2(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{|N_j^T|}
\end{aligned} \tag{24}
$$

At this point, an expression for $\overrightarrow{\bigtriangledown}\theta_j^T$ has been provided. To complete the expression of Jm^T, Eq. 24 and 18 need to be combined:

$$
\begin{aligned}
Jm^T =& \sum_j \frac{\cos(\theta_j^T)\sin(\theta_j^T) N_j^T \cdot (J{N_j^T}^t \cdot N_j^T)^t}{2|N_j^T|^3} \\
&- \sum_j \frac{\sin(\theta_j^T)^2 N_j^T \cdot (2\eta - p_{t_{j+2}} - p_{t_{j+1}})^t}{2|N_j^T|^2} \\
&+ \sum_j \frac{\theta_j^T J N_j^T}{2|N_j^T|} \\
&- \sum_j \frac{\theta_j^T N_j^T \cdot (J{N_j^T}^t \cdot N_j^T)^t}{2|N_j^T|^3}
\end{aligned}
$$

Final expression of Jm^T

We obtain the final expression of $Jm^T(\eta)$ as:

$$
\begin{aligned}
Jm^T = \quad & \sum_j \frac{e_1(\theta_j^T) N_j^T \cdot {N_j^T}^t \cdot J N_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
& - \sum_j \frac{N_j^T \cdot (2\eta - p_{t_{j+1}} - p_{t_{j+2}})^t}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} \\
& + \sum_j \frac{e_2(\theta_j^T) J N_j^T}{2|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|}
\end{aligned} \tag{25}
$$

where $e_1(x) = \frac{\cos(x)\sin(x) - x}{\sin(x)^3}$ and $e_2(x) = \frac{x}{\sin(x)}$.

Given the final expression of Jm^T (see Eq. 25), we recall that $\vec{\bigtriangledown} w_{t_i}^T$ can be computed with the following expression: $\vec{\bigtriangledown} w_{t_i}^T = \frac{{B^T}^t \cdot N_i^T}{\det(A^T)}$ (with $B^T = Jm^T + \sum_j w_{t_j}^T \cdot I_3$).

4.2 Special case: $\eta \in Support(T), \notin T$

The expressions provided so far admit degenerate cases when $\det(A^T) = 0$ (see Eq. 17). Similarly to the 2D setting, these cases only occur when η is lying on the support plane of T, noted $Support(T)$.

For $\eta \in Support(T), \notin T$, as discussed in [Ju et al.(2005)Ju, Schaefer, and Warren], given small steps in the support of T, the weights are set to 0 (since $\eta + d\eta \in Support(T)$): $\forall(\eta + d\eta) \in Support(T), \notin T$, $w_{t_i}^T(\eta + d\eta) = 0$.

Therefore, the weights only evolve in the direction of the normal of T:

$$
\forall \eta \in Support(T), \notin T: \quad \vec{\bigtriangledown} w_{t_i}^T = \frac{\partial w_{t_i}^T(\eta + \epsilon n_T)}{\partial \epsilon}_{|\epsilon \to 0} \cdot n_T
$$

To evaluate the above expression, we consider the Taylor expansion of $w_{t_i}^T(\eta + \epsilon n_T) = \frac{N_i^T(\eta + \epsilon n_T)^t \cdot m^T(\eta + \epsilon n_T)}{\det(A^T(\eta + \epsilon n_T))}$ with respect to ϵ.

The details of this expansion can be found in section 7. The final result is given by:

$$
\begin{aligned}
\vec{\bigtriangledown} w_{t_i}^T = & -\sum_j \frac{e_2(\theta_j^T)(p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}})}{4|T||p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|} n_T \\
& - \sum_j \frac{e_1(\theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2 {N_i^T}^t \cdot N_j^T}{8|T|(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} n_T \\
& + \sum_j \frac{{N_i^T}^t \cdot N_j^T}{4|T|(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} n_T \\
& \forall \eta \in Support(T), \notin T
\end{aligned} \tag{26}
$$

4.3 Expression of the MV-Hessians

We note $\delta^x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \delta^y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \delta^z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Derivation of $Hw_{t_i}^T$

By differentiating Eq. 16 successively by $c = x, y$, and z, we obtain:

$$\sum_i \partial_c(p_{t_i} - \eta) \cdot \overrightarrow{\bigtriangledown} w_i^{T^t} + \sum_i (p_{t_i} - \eta) \cdot \partial_c(\overrightarrow{\bigtriangledown} w_i^{T^t}) = \partial_c(Jm^T) + \sum_i \partial_c(w_i^T) \cdot I_3$$

Therefore

$$\sum_j (p_{t_j} - \eta) \cdot \partial_c(\overrightarrow{\bigtriangledown} w_{t_j}^{T^t}) = C_c^T \tag{27}$$

with $C_c^T \triangleq \delta^c \cdot \sum_j \overrightarrow{\bigtriangledown} w_{t_j}^{T^t} + \partial_c(Jm^T) + \sum_j \partial_c(w_{t_j}^T) \cdot I_3$.

All these terms have been expressed previously, except for the derivatives of Jm^T.

Expression of the derivatives of Jm^T

Given a vector $k \in \mathbb{R}^3$, we note $(k)_{(x)}, (k)_{(y)}, (k)_{(z)}$ its components in x, y, and z.

By differentiating Eq. 25 term by term, and using Eq. 24 for $\overrightarrow{\bigtriangledown}\theta_j^T$, we obtain the final expression of $\partial_c(Jm^T)$

$$\begin{aligned}
\partial_c(Jm^T) = &\sum_j \frac{e_3(\theta_j^T)(JN_j^{T^t} \cdot N_j^T)_{(c)} N_j^T \cdot N_j^{T^t} \cdot JN_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^5} - \sum_j \frac{e_4(\theta_j^T)(2\eta - p_{t_{j+1}} - p_{t_{j+2}})_{(c)} N_j^T \cdot N_j^{T^t} \cdot JN_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^4} \\
&+ \sum_j \frac{e_1(\theta_j^T)\partial_c(N_j^T) \cdot N_j^{T^t} \cdot JN_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} + \sum_j \frac{e_1(\theta_j^T) N_j^T \cdot \partial_c(N_j^T)^t \cdot JN_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
&- \sum_j \frac{3e_1(\theta_j^T)(\eta - p_{t_{j+1}})_{(c)} N_j^T \cdot N_j^{T^t} \cdot JN_j^T}{2|p_{t_{j+2}} - \eta|^3 |p_{t_{j+1}} - \eta|^5} - \sum_j \frac{3e_1(\theta_j^T)(\eta - p_{t_{j+2}})_{(c)} N_j^T \cdot N_j^{T^t} \cdot JN_j^T}{2|p_{t_{j+2}} - \eta|^5 |p_{t_{j+1}} - \eta|^3} \\
&- \sum_j \frac{\partial_c(N_j^T) \cdot (2\eta - p_{t_{j+1}} - p_{t_{j+2}})^t}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} + \sum_j \frac{(\eta - p_{t_{j+1}})_{(c)} N_j^T \cdot (2\eta - p_{t_{j+1}} - p_{t_{j+2}})^t}{|p_{t_{j+2}} - \eta|^2 |p_{t_{j+1}} - \eta|^4} \\
&+ \sum_j \frac{(\eta - p_{t_{j+2}})_{(c)} N_j^T \cdot (2\eta - p_{t_{j+1}} - p_{t_{j+2}})^t}{|p_{t_{j+2}} - \eta|^4 |p_{t_{j+1}} - \eta|^2} + \sum_j \frac{e_5(\theta_j^T)(JN_j^{T^t} \cdot N_j^T)_{(c)} JN_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
&- \sum_j \frac{e_6(\theta_j^T)(2\eta - p_{t_{j+1}} - p_{t_{j+2}})_{(c)} JN_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} - \sum_j \frac{(\eta - p_{t_{j+1}})_{(c)} e_2(\theta_j^T) JN_j^T}{2|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|^3} \\
&- \sum_j \frac{(\eta - p_{t_{j+2}})_{(c)} e_2(\theta_j^T) JN_j^T}{2|p_{t_{j+2}} - \eta|^3 |p_{t_{j+1}} - \eta|} - \sum_j \frac{N_j^T \cdot \delta^{ct}}{|p_{t_{j+2}} - \eta|^2 |p_{t_{j+1}} - \eta|^2}
\end{aligned}$$

where $e_3(x) = (3cos(x)(cos(x)x - sin(x)) + cos(x)sin(x)^3)/sin(x)^5$, $e_4(x) = (3(cos(x)x - sin(x)) + sin(x)^3)/sin(x)^3$, $e_5(x) = (sin(x) - xcos(x))cos(x)/sin(x)^3$, and $e_6(x) = (sin(x) - xcos(x))/sin(x)$.

As previously discussed, $|N_j^T|$ can become close to 0 only if θ_j^T tends to 0 or π.

The first case corresponds to η lying on the same line as one edge of the triangle, but not **on** the edge. All the functions $e_i(x)$ in this expression admit well-defined Taylor expansions, given in section 6. As these expressions are shown to converge, they provide a practical way to robustly evaluate $\partial_c(Jm^T)$ near the support lines of the edges of the cage triangles.

The second case corresponds to η lying on one edge $[p_{t_{j+1}} p_{t_{j+2}}]$ of the triangle T. As discussed in Sec. 2, we do not provide expressions of the derivatives in that case (for points lying directly on the triangle T).

When η lies exactly onto the support plane of a triangle T, we cannot use the same strategy to compute the Hessians. These special cases will be discussed in the next paragraph.

Final expression of $Hw_{t_i}^T$

From Eq. 27, once again we obtain

$$\left\{ \begin{array}{ll} \partial_x(\overrightarrow{\bigtriangledown} w_{t_i}^T) & = \frac{C_x^{T^t} \cdot N_i^T}{\det(A^T)} \\ \partial_y(\overrightarrow{\bigtriangledown} w_{t_i}^T) & = \frac{C_y^{T^t} \cdot N_i^T}{\det(A^T)} \\ \partial_z(\overrightarrow{\bigtriangledown} w_{t_i}^T) & = \frac{C_z^{T^t} \cdot N_i^T}{\det(A^T)} \end{array} \right\} \quad \forall \eta \notin Support(T).$$

Since

$$Hw_{t_i}^T = \begin{pmatrix} \partial_x(\overrightarrow{\bigtriangledown} w_{t_i}^T)^t \\ \partial_y(\overrightarrow{\bigtriangledown} w_{t_i}^T)^t \\ \partial_z(\overrightarrow{\bigtriangledown} w_{t_i}^T)^t \end{pmatrix}$$

finally we obtain

$$Hw_{t_i}^T = \frac{1}{\det(A^T)} \begin{pmatrix} {N_i^T}^t \cdot \partial_x(Jm^T) \\ {N_i^T}^t \cdot \partial_y(Jm^T) \\ {N_i^T}^t \cdot \partial_z(Jm^T) \end{pmatrix} + \frac{1}{\det(A^T)}(N_i^T \cdot (\sum_j \overrightarrow{\bigtriangledown} w_j^T)^t + \sum_j \overrightarrow{\bigtriangledown} w_j^T \cdot {N_i^T}^t) \quad \forall \eta \notin Support(T) \tag{28}$$

4.4 Special case: $\eta \in Support(T), \notin T$

Following the same strategy than in Sec. 3.5 for the 2D case, we derive the value of the Hessian in the particular case of $\eta \in Support(T), \notin T$ as:

$$Hw_{t_i}^T = \overrightarrow{\bigtriangledown} dw_i^T \cdot n_T^t + n_T \cdot \overrightarrow{\bigtriangledown} d{w_i^T}^t \quad \forall \eta \in Support(T), \notin T \tag{29}$$

The details of the derivation of $\overrightarrow{\bigtriangledown} dw_i^T$ can be found in section 8 The final expression of $\overrightarrow{\bigtriangledown} dw_i^T$ is given by:

$$\begin{aligned}
-2|T|\overrightarrow{\bigtriangledown} dw_i^T = &\sum_j \frac{e_1(\theta_j^T)((p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}}))J{N_j^T}^t \cdot N_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
&- \sum_j \frac{((p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}}))(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} \\
&- \sum_j \frac{e_7(\theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2({N_i^T}^t \cdot N_j^T)J{N_j^T}^t \cdot N_j^T}{4(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^5} \\
&+ \sum_j \frac{|p_{t_{j+2}} - p_{t_{j+1}}|^2({N_i^T}^t \cdot N_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^4} \\
&+ \sum_j \frac{e_1(\theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2(J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{4(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
&+ \sum_j \frac{({N_i^T}^t \cdot N_j^T)J{N_j^T}^t \cdot N_j^T}{(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^4} \\
&+ \sum_j \frac{\cos(\theta_j^T)({N_i^T}^t \cdot N_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
&- \sum_j \frac{(J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2}
\end{aligned}$$

where $e_7(x) = \frac{2\cos(x)\sin(x)^3 + 3(\sin(x)\cos(x) - x)}{\sin(x)^5}$.

5 Continuity between the general case and the special case

We obtained the formulae for the gradient and the Hessian of $w_i^T(\eta)$ in the general case, when the point of interest η does not lie on the triangle T, and in the special case when η lies on it.

As MVC are C^∞ everywhere not on M, these formulae are guaranteed to converge, since in particular, the gradient and the Hessian are continuous functions everywhere not on M.

The same holds in 2D where the distinction is made for the computation of $w_i^E(\eta)$ whether η lies on the line supported by the edge E or not.

6 Taylor expansion formulae of e_i

In the paper, we refer the reader to several functions noted $e_i(x)$. We give here their corresponding Taylor expansion formulae that we obtained using Mapple. We note " $=_{\{0\}}$ " the equality of the equivalent around 0.

$$\begin{aligned}
e_1(x) =& \frac{\cos(x)\sin(x) - x}{\sin(x)^3} \\
=_{\{0\}}& -\frac{2}{3} - \frac{1}{5}x^2 - \frac{17}{420}x^4 - \frac{29}{4200}x^6 - \frac{1181}{1108800}x^8 - \frac{1393481}{9081072000}x^{10} \\
& - \frac{763967}{36324288000}x^{12} - \frac{133541}{48117888000}x^{14} - \frac{3821869001}{10751460894720000}x^{16} \\
& - \frac{115665628927}{2601853536522240000}x^{18} - \frac{8388993163723}{153881052017172480000}x^{20} - \frac{3868248770144093}{588133380809633218560000}x^{22} \\
& - \frac{3682368472021807}{4705067046477065748480000}x^{24} - \frac{269101073327718589}{2923863093167890857984000000}x^{26} + O\left(x^{27}\right)
\end{aligned} \tag{30}$$

$$\begin{aligned}
e_2(x) =& \frac{x}{\sin(x)} \\
=_{\{0\}}& 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \frac{127}{604800}x^8 + \frac{73}{3421440}x^{10} + \frac{1414477}{653837184000}x^{12} \\
& + \frac{8191}{37362124800}x^{14} + \frac{16931177}{762187345920000}x^{16} + \frac{5749691557}{2554547108585472000}x^{18} + \frac{91546277357}{401428831349145600000}x^{20} \\
& + \frac{3324754717}{143888775912161280000}x^{22} + \frac{1982765468311237}{846912068365871834726400000}x^{24} \\
& + \frac{22076500342261}{93067260259985915904000000}x^{26} + \frac{65053034220152267}{2706661834818276108533760000000}x^{28} + O\left(x^{30}\right)
\end{aligned} \tag{31}$$

$$\begin{aligned}
e_3(x) =& \frac{3\cos(x)(\cos(x)x - \sin(x)) + \cos(x)\sin(x)^3}{\sin(x)^5} \\
=_{\{0\}}& -\frac{2}{5} - \frac{1}{35}x^2 + \frac{3}{140}x^4 + \frac{1349}{138600}x^6 + \frac{267767}{100900800}x^8 + \frac{1752539}{3027024000}x^{10} \\
& + \frac{204708709}{1852538688000}x^{12} + \frac{862222247}{44797753728000}x^{14} + \frac{1359125231539}{433642256087040000}x^{16} + \frac{260976933802873}{538583682060103680000}x^{18} \\
& + \frac{529743964972219}{7370092491348787200000}x^{20} + \frac{118166997202277}{11464588319875891200000}x^{22} \\
& + \frac{5881774875406174993}{4093408330435047201177600000}x^{24} + O\left(x^{26}\right)
\end{aligned} \tag{32}$$

$$\begin{aligned}
e_4(x) =& \frac{3(\cos(x)x - \sin(x)) + \sin(x)^3}{\sin(x)^3} \\
=_{\{0\}}& -\frac{2}{5}x^2 - \frac{2}{21}x^4 - \frac{4}{225}x^6 - \frac{2}{693}x^8 - \frac{2764}{6449625}x^{10} - \frac{4}{66825}x^{12} \\
& - \frac{28936}{3618239625}x^{14} - \frac{87734}{84922212375}x^{16} - \frac{698444}{5373085843125}x^{18} - \frac{310732}{19405276970625}x^{20} \\
& - \frac{1890912728}{975456869390128125}x^{22} - \frac{2631724}{11378955872851875}x^{24} - \frac{27142241176}{995861834515125703125}x^{26} + O\left(x^{28}\right)
\end{aligned} \tag{33}$$

$$\begin{aligned}
e_5(x) =& \frac{(\sin(x) - x\cos(x))\cos(x)}{\sin(x)^3} \\
=_{\{0\}}& \frac{1}{3} - \frac{1}{30}x^2 - \frac{53}{2520}x^4 - \frac{367}{75600}x^6 - \frac{5689}{6652800}x^8 - \frac{7198361}{54486432000}x^{10} \\
& - \frac{1121539}{59439744000}x^{12} - \frac{8117471}{3175780608000}x^{14} - \frac{236480428279}{709596419051520000}x^{16} - \frac{5929710926423}{140500090972200960000}x^{18} \\
& - \frac{48228394603127}{9232863121030348800000}x^{20} - \frac{22394113150293893}{35288002848577993113600000}x^{22} \\
& - \frac{531403859736209}{6999273292279932518400000}x^{24} - \frac{1572992236821149549}{17543178559007345147904000000}x^{26} + O\left(x^{28}\right)
\end{aligned} \tag{34}$$

$$
\begin{aligned}
e_6(x) =&\frac{\sin(x) - x\cos(x)}{\sin(x)}\\
=_{\{0\}}&\frac{1}{3}x^2 + \frac{1}{45}x^4 + \frac{2}{945}x^6 + \frac{1}{4725}x^8 + \frac{2}{93555}x^{10} + \frac{1382}{638512875}x^{12}\\
&+ \frac{4}{18243225}x^{14} + \frac{3617}{162820783125}x^{16} + \frac{87734}{38979295480125}x^{18} + \frac{349222}{1531329465290625}x^{20}\\
&+ \frac{310732}{13447856940643125}x^{22} + \frac{472728182}{201919571963756521875}x^{24} + \frac{2631724}{1109448197603057812 5}x^{26}\\
&+ \frac{13571120588}{564653660170076273671875}x^{28} + O\left(x^{30}\right)
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
e_7(x) =&\frac{2\cos(x)\sin(x)^3 + 3(\sin(x)\cos(x) - x)}{\sin(x)^5}\\
=_{\{0\}}& - \frac{8}{5} - \frac{4}{7}x^2 - \frac{1}{7}x^4 - \frac{211}{6930}x^6 - \frac{29509}{5045040}x^8 - \frac{157301}{151351200}x^{10} - \frac{16079783}{92626934400}x^{12}\\
&- \frac{61760113}{2239887686400}x^{14} - \frac{30359523011}{7227370934784000}x^{16} - \frac{16640264468327}{26929184103005184000}x^{18} - \frac{617766523408427}{70015878667813478400 00}x^{20}\\
&- \frac{109430933144243}{891111183044898816 0000}x^{22} - \frac{342143652330193331}{204670416521752360058880000}x^{24} + O\left(x^{25}\right)
\end{aligned}
\tag{36}
$$

Discussion The Taylor expansions of these functions may not give directly a result with the expected error (i. e. of order x^N) when computing them in this straightforward manner. This is caused by floating point imprecision. Still, they can be approximated with any precision using look-up tables, e. g. using an *infinite precision* library[1].

[1] For instance, see the GNU Multiple Precision Arithmetic Library, http://gmplib.org/

7 Taylor expansion of $w^T_{t_i}(\eta + \epsilon n_T)$

We present here the details for the Taylor expansion to the order 1 of $w^T_{t_i}(\eta+\epsilon n_T) = \frac{N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T)}{\det(A^T(\eta+\epsilon n_T))}$, in the special case where $\eta \in Support(T), \notin T$.

The determinant of a basis of $\mathbb{R}^3$ is the volume of its generated parallelepiped, and the following expression holds:

$$\det(A^T(\eta+\epsilon n_T)) = -2\epsilon \cdot |T| \tag{37}$$

where $|T|$ is the area of T. All that is left is to write the Taylor expansion of $N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T)$ to the order 2.

$$\begin{aligned}N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T) =& \frac{1}{2}\sum_j \theta_j(\eta+\epsilon n_T)\cdot N_i^T(\eta+\epsilon n_T)^t \cdot n_j^T(\eta+\epsilon n_T)\\ =& \frac{1}{2}\sum_j \theta_j(\eta+\epsilon n_T)\cdot N_i^T(\eta+\epsilon n_T)^t \cdot \frac{N_j^T(\eta+\epsilon n_T)}{|N_j^T(\eta+\epsilon n_T)|}\end{aligned}$$

We develop $\theta_j^T(\eta+\epsilon n_T)$ to the second order:

$$\theta_j^T(\eta+\epsilon n_T) = \theta_j + \epsilon \vec{\nabla}\theta_j^{T^t} \cdot n_T + \frac{\epsilon^2}{2} n_T^t \cdot H\theta_j^T \cdot n_T + o(\epsilon^2)$$

From the expression of $\vec{\nabla} C_j^T$, we know that $\vec{\nabla} C_j^{T^t} \cdot n_T = 0 \;\forall\, \eta \in Support(T)$. Therefore, $\vec{\nabla}\theta_j^{T^t} \cdot n_T = 0 \;\forall\, \eta \in Support(T)$, and

$$\theta_j^T(\eta+\epsilon n_T) = \theta_j + \frac{\epsilon^2}{2} n_T^t \cdot H\theta_j^T \cdot n_T + o(\epsilon^2) \tag{38}$$

We develop $|N_j^T(\eta+\epsilon n_T)|^{-1}$ to the second order:

$$\begin{aligned}\frac{1}{|N_j^T(\eta+\epsilon n_T)|} =& \frac{1}{|N_j^T|} - \epsilon\frac{(JN_j^{T^t}\cdot N_j^T)^t \cdot n_T}{|N_j^T|^3} - \frac{\epsilon^2}{2}\frac{n_T^t \cdot (JN_j^{T^t} \cdot JN_j^T)\cdot n_T}{|N_j^T|^3}\\ &+ 3\frac{\epsilon^2}{2}\frac{n_T^t \cdot (JN_j^{T^t}\cdot N_j^T \cdot N_j^{T^t} \cdot JN_j^T)\cdot n_T}{|N_j^T|^5} + o(\epsilon^2)\end{aligned}$$

We can start simplifying this expression by noticing that $(JN_j^T \cdot N_j^T)^t \cdot n_T = 0$ and $n_T^t \cdot (JN_j^{T^t} \cdot JN_j^T)\cdot n_T = |JN_j^T \cdot n_T|^2 = |p_{t_{j+2}} - p_{t_{j+1}}|^2$.

Therefore the development of the function $\eta \to |N_j^T|^{-1}$ in the direction of n_T is equal to

$$\frac{1}{|N_j^T(\eta+\epsilon n_T)|} = \frac{1}{|N_j^T|} - \frac{\epsilon^2}{2}\frac{|p_{t_{i+2}} - p_{t_{i+1}}|^2}{|N_j^T|^3} + o(\epsilon^2) \tag{39}$$

It leads to the following development to the order 2 of $N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T)$:

$$\begin{aligned}N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T) =& \frac{1}{2}(N_i^T + \epsilon JN_i^T\cdot n_T)^t \cdot \sum_j (N_j^T + \epsilon JN_j^T \cdot n_T)(\theta_j + \frac{\epsilon^2}{2}n_T^t \cdot H\theta_j^T \cdot n_T)(\frac{1}{|N_j^T|} - \frac{\epsilon^2}{2}\frac{|p_{t_{i+2}} - p_{t_{i+1}}|^2}{|N_j^T|^3})\\ &+ o(\epsilon^2)\end{aligned}$$

We note that $(JN_i \cdot n_T)^t \cdot N_j = 0$ and $N_i^t \cdot (JN_j \cdot n_T) = 0$, since N_k and n_T are colinear $\forall k, \quad \forall \eta \in Support(T)$.

We also note that

$$\begin{aligned}(JN_i \cdot n_T)^t \cdot (JN_j \cdot n_T) &= ((p_{t_{i+2}} - p_{t_{i+1}}) \wedge n_T)^t \cdot ((p_{t_{j+2}} - p_{t_{j+1}}) \wedge n_T)\\ &= (p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}})\end{aligned}$$

By developing all the terms, we see that all that is left is the order two (plus higher orders, that we do not write, since they are negligeable in front of ϵ^2).

$$\begin{aligned}N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T) =& \epsilon^2 \sum_j \frac{\theta_j^T (p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}})}{2|N_j^T|}\\ &+ \epsilon^2 \sum_j \frac{n_T^t \cdot H\theta_j^T \cdot n_T \cdot N_i^{T^t} \cdot N_j^T}{4|N_j^T|}\\ &- \epsilon^2 \sum_j \frac{\theta_j^T |p_{t_{j+2}} - p_{t_{j+1}}|^2 N_i^{T^t} \cdot N_j^T}{4|N_j^T|^3} + o(\epsilon^2)\end{aligned}$$

$$N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T) = \begin{aligned} &\epsilon^2 \textstyle\sum_j \frac{\theta_j^T (p_{t_{i+2}}-p_{t_{i+1}})^t \cdot (p_{t_{j+2}}-p_{t_{j+1}})}{2|N_j^T|} \\ &+\epsilon^2 \textstyle\sum_j \frac{(|N_j^T|^2 {n_T}^t \cdot H\theta_j^T \cdot n_T - \theta_j^T |p_{t_{j+2}}-p_{t_{j+1}}|^2) \cdot {N_i^T}^t \cdot N_j^T}{4|N_j^T|^3} \\ &+o(\epsilon^2) \end{aligned} \tag{40}$$

This expression is not well defined when $\exists j \ / \ \theta_j^T = 0$. In fact, $|N_j^T| \sim \theta_j^T$ around 0, and therefore the term under the first sum tends to a constant when $\theta_j^T \to 0$.

By combining this expression with the one of $H\theta_j^T$, we will obtain the final expression of $\overrightarrow{\bigtriangledown} w_i^T$, and we will be able to prove that the j-th term under the second sum tends to 0 when $\theta_j^T \to 0$.

By differentiating twice $C_j^T = \cos(\theta_j^T) = \frac{<p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>}{|p_{t_{j+1}}-\eta||p_{t_{j+2}}-\eta|}$, we have that $-\cos(\theta_j^T) \cdot \overrightarrow{\bigtriangledown}\theta_j^T \cdot {\overrightarrow{\bigtriangledown}\theta_j^T}^t - \sin(\theta_j^T) H\theta_j^T = HC_j^T$, with

$$\overrightarrow{\bigtriangledown} C_j^T = \frac{2\eta - p_{t_{j+1}} - p_{t_{j+2}}}{|p_{t_{j+2}}-\eta| \cdot |p_{t_{j+1}}-\eta|} - \frac{<p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>(\eta - p_{t_{j+2}})}{|p_{t_{j+2}}-\eta|^3 |p_{t_{j+1}}-\eta|} - \frac{<p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>(\eta - p_{t_{j+1}})}{|p_{t_{j+1}}-\eta|^3 |p_{t_{j+2}}-\eta|} \tag{41}$$

and

$$\begin{aligned} HC_j^T =& \frac{2I_3}{|p_{t_{j+2}}-\eta| \cdot |p_{t_{j+1}}-\eta|} - \frac{(2\eta - p_{t_{j+1}} - p_{t_{j+2}}) \cdot (\eta - p_{t_{j+1}})^t}{|p_{t_{j+2}}-\eta| \cdot |p_{t_{j+1}}-\eta|^3} - \frac{(2\eta - p_{t_{j+1}} - p_{t_{j+2}}) \cdot (\eta - p_{t_{j+2}})^t}{|p_{t_{j+2}}-\eta|^3 \cdot |p_{t_{j+1}}-\eta|} \\ &- \frac{I_3 <p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>}{|p_{t_{j+2}}-\eta|^3 |p_{t_{j+1}}-\eta|} - \frac{(\eta - p_{t_{j+2}}) \cdot (2\eta - p_{t_{j+1}} - p_{t_{j+2}})^t}{|p_{t_{j+2}}-\eta|^3 |p_{t_{j+1}}-\eta|} \\ &+ \frac{<p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>(\eta - p_{t_{j+2}}) \cdot (\eta - p_{t_{j+1}})^t}{|p_{t_{j+2}}-\eta|^3 |p_{t_{j+1}}-\eta|^3} + \frac{3<p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>(\eta - p_{t_{j+2}}) \cdot (\eta - p_{t_{j+2}})^t}{|p_{t_{j+2}}-\eta|^5 |p_{t_{j+1}}-\eta|} \\ &- \frac{I_3 <p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>}{|p_{t_{j+1}}-\eta|^3 |p_{t_{j+2}}-\eta|} - \frac{(\eta - p_{t_{j+1}}) \cdot (2\eta - p_{t_{j+1}} - p_{t_{j+2}})^t}{|p_{t_{j+1}}-\eta|^3 |p_{t_{j+2}}-\eta|} \\ &+ \frac{<p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>(\eta - p_{t_{j+1}}) \cdot (\eta - p_{t_{j+2}})^t}{|p_{t_{j+1}}-\eta|^3 |p_{t_{j+2}}-\eta|^3} + \frac{3<p_{t_{j+1}}-\eta|p_{t_{j+2}}-\eta>(\eta - p_{t_{j+1}}) \cdot (\eta - p_{t_{j+1}})^t}{|p_{t_{j+1}}-\eta|^5 |p_{t_{j+2}}-\eta|} \end{aligned} \tag{42}$$

Since $(p_{t_k} - \eta)^t \cdot n_T = 0 \quad \forall k, \forall \eta \in Support(T)$, we have that ${\overrightarrow{\bigtriangledown}\theta_j^T}^t \cdot n_T = 0$ and we obtain:

$$\begin{aligned} {n_T}^t \cdot H\theta_j^T \cdot n_T =& \frac{-{n_T}^t \cdot HC_j^T \cdot n_T}{\sin(\theta_j^T)} \\ =& \frac{\cos(\theta_j^T)}{\sin(\theta_j^T)|p_{t_{j+2}}-\eta|^2} + \frac{\cos(\theta_j^T)}{\sin(\theta_j^T)|p_{t_{j+1}}-\eta|^2} - \frac{2}{\sin(\theta_j^T)|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|} \end{aligned} \tag{43}$$

We now focus on the expression of $|N_j^T|^2 {n_T}^t \cdot H\theta_j^T \cdot n_T - \theta_j^T |p_{t_{j+2}} - p_{t_{j+1}}|^2$ that appears in Eq. 40:

$$\begin{aligned} &|N_j^T|^2 {n_T}^t \cdot H\theta_j^T \cdot n_T - \theta_j^T |p_{t_{j+2}} - p_{t_{j+1}}|^2 \\ &\quad = \frac{|N_j^T|^2 \sin(\theta_j^T)(\cos(\theta_j^T)(|p_{t_{j+2}}-\eta|^2 + |p_{t_{j+2}}-\eta|^2) - 2|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)}{\sin(\theta_j^T)^2 |p_{t_{j+2}}-\eta|^2 |p_{t_{j+1}}-\eta|^2} - \theta_j^T |p_{t_{j+2}} - p_{t_{j+1}}|^2 \\ &\quad = \sin(\theta_j^T)(\cos(\theta_j^T)(|p_{t_{j+2}}-\eta|^2 + |p_{t_{j+2}}-\eta|^2) - 2|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|) - \theta_j^T |p_{t_{j+2}} - p_{t_{j+1}}|^2 \end{aligned}$$

Since $|p_{t_{j+2}} - p_{t_{j+1}}|^2 = |p_{t_{j+2}}-\eta|^2 + |p_{t_{j+1}}-\eta|^2 - 2\cos(\theta_j^T)|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|$, by substituting $|p_{t_{j+2}}-\eta|^2 + |p_{t_{j+1}}-\eta|^2$ in the previous expression, we have

$$\begin{aligned} &|N_j^T|^2 {n_T}^t \cdot H\theta_j^T \cdot n_T - \theta_j^T |p_{t_{j+2}} - p_{t_{j+1}}|^2 \\ &\quad = \sin(\theta_j^T)\cos(\theta_j^T)(|p_{t_{j+2}} - p_{t_{j+1}}|^2 + 2\cos(\theta_j^T)|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta| - 2|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)v - \theta_j^T |p_{t_{j+2}} - p_{t_{j+1}}|^2 \\ &\quad = (\sin(\theta_j^T)\cos(\theta_j^T) - \theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2 + 2(\cos^2(\theta_j^T) - 1)|N_j^T| \\ &\quad = (\sin(\theta_j^T)\cos(\theta_j^T) - \theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2 - 2\sin^2(\theta_j^T)|N_j^T| \end{aligned}$$

This expression is an equivalent of ${\theta_j^T}^3$ around 0, which proves that the problematic term under the second sum in the expression of $N_i^T(\eta+\epsilon n_T)^t \cdot m^T(\eta+\epsilon n_T)$ tends to 0, and can be neglected in the computation of $\overrightarrow{\bigtriangledown} w_{t_i}^T$ in the special case of $\eta \in Support(T), \notin T$.

Finally, $\forall \eta \in Support(T), \notin T$:

$$\overrightarrow{\bigtriangledown} w_{t_i}^T = dw_i^T n_T \tag{44}$$

with

$$\begin{aligned} -2|T|dw_i^T = \quad & \sum_j \frac{e_2(\theta_j^T)(p_{t_{i+2}}-p_{t_{i+1}})^t \cdot (p_{t_{j+2}}-p_{t_{j+1}})}{2|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|} \\ & + \sum_j \frac{e_1(\theta_j^T)|p_{t_{j+2}}-p_{t_{j+1}}|^2 N_i^{T\,t} \cdot N_j^T}{4(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^3} \\ & - \sum_j \frac{N_i^{T\,t} \cdot N_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^2} \end{aligned} \tag{45}$$

8 Expression of the Hessian in 3D, in the case of alignment with the triangle T

We have seen that if $\eta \in Support(T), \notin T$, the gradient of the unnormalized weight with respect to T $w_{t_i}^T(\eta)$ is given by $\overrightarrow{\bigtriangledown} w_{t_i}^T(\eta) = dw_i^T(\eta) n_T$, with $dw_i^T(\eta)$ a scalar function whose form was presented previously in the paper (see Eq. 45). In order to get the Hessian of the unnormalized weight with respect to T, we need to differentiate dw_i^T since $Hw_{t_i}^T = \overrightarrow{\bigtriangledown} dw_i^T \cdot n_T^t + n_T \cdot \overrightarrow{\bigtriangledown} dw_i^{T^t}$ (and not simply $Hw_{t_i}^T = \overrightarrow{\bigtriangledown} dw_i^T \cdot n_T^t$, see discussion Sec. 3.5).

Expression of $\overrightarrow{\bigtriangledown} dw_i^T$ We recall that we obtained the expression of $dw_i^T \equiv \frac{\partial (w_{t_i}^T(\eta + \epsilon n_T))}{\partial \epsilon}_{|\epsilon \to 0}$ as

$$
\begin{aligned}
-2|T| dw_i^T = & \sum_j \frac{e_2(\theta_j^T)(p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}})}{2|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|} \\
& + \sum_j \frac{e_1(\theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2 N_i^{T^t} \cdot N_j^T}{4(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
& - \sum_j \frac{N_i^{T^t} \cdot N_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2}
\end{aligned}
$$

We need to differentiate this expression in order to compute $\overrightarrow{\bigtriangledown} dw_i^T$. We chose to express it by putting back the various terms $\sin^k(\theta_j^T)$ in the fractions before differentiating, so that only terms of the form $|N_j^T|^k$ remain as the denominator of the fractions. We also note $u_{km} = (p_{t_{k+2}} - p_{t_{k+1}})^t \cdot (p_{t_{m+2}} - p_{t_{m+1}})$ to simplify this expression. We obtain

$$
-2|T| dw_i^T = \sum_j \frac{\theta_j^T u_{ij}}{2|N_j^T|} + \sum_j \frac{\bar{e_1}(\theta_j^T) u_{jj} N_i^{T^t} \cdot N_j^T}{4|N_j^T|^3} - \sum_j \frac{\sin^2(\theta_j^T) N_i^{T^t} \cdot N_j^T}{2|N_j^T|^2} \tag{46}
$$

with $\bar{e_1}(x) = \cos(x)\sin(x) - x$ (and therefore $\frac{d\bar{e_1}}{dx}(x) = -2\sin^2(x)$).

By differentiating Eq. 46, we obtain that

$$
\begin{aligned}
-2|T| \overrightarrow{\bigtriangledown} dw_i^T = & \sum_j \frac{u_{ij} \overrightarrow{\bigtriangledown} \theta_j^T}{2|N_j^T|} [A] \\
& - \sum_j \frac{\theta_j^T u_{ij} J N_j^{T^t} \cdot N_j^T}{2|N_j^T|^3} [B] \\
& - \sum_j \frac{\sin(\theta_j^T)^2 u_{jj} (N_i^{T^t} \cdot N_j^T) \overrightarrow{\bigtriangledown} \theta_j^T}{2|N_j^T|^3} [C] \\
& + \sum_j \frac{\bar{e_1}(\theta_j^T) u_{jj} (J N_j^{T^t} \cdot N_i^T + J N_i^{T^t} \cdot N_j^T)}{4|N_j^T|^3} [D] \\
& - \sum_j \frac{3\bar{e_1}(\theta_j^T) u_{jj} (N_i^{T^t} \cdot N_j^T) J N_j^{T^t} \cdot N_j^T}{4|N_j^T|^5} [E] \\
& - \sum_j \frac{\sin(\theta_j^T)\cos(\theta_j^T)(N_i^{T^t} \cdot N_j^T) \overrightarrow{\bigtriangledown} \theta_j^T}{|N_j^T|^2} [F] \\
& - \sum_j \frac{\sin^2(\theta_j^T)(J N_j^{T^t} \cdot N_i^T + J N_i^{T^t} \cdot N_j^T)}{2|N_j^T|^2} [G] \\
& + \sum_j \frac{\sin^2(\theta_j^T)(N_i^{T^t} \cdot N_j^T) J N_j^{T^t} \cdot N_j^T}{|N_j^T|^4} [H]
\end{aligned}
$$

Here we put a letter $[X]$ to help the reader follow through the differentiation.

By replacing $\overrightarrow{\bigtriangledown}\theta_j^T$ by the expression given in Eq. 24, we have ([X1] – resp [X2] – indicates that the part of the equation came from previous equation [X] when replacing $\overrightarrow{\bigtriangledown}\theta_j^T$ by the first (i. e. $\frac{\cos(\theta_j^T)\sin(\theta_j^T)J{N_j^T}^t\cdot N_j^T}{|N_j^T|^2}$) – resp second (i. e. $-\frac{\sin(\theta_j^T)^2(2\eta-p_{t_{j+2}}-p_{t_{j+1}})}{|N_j^T|}$) – part of its expression)

$$
\begin{aligned}
-2|T|\overrightarrow{\bigtriangledown}dw_i^T = &\sum_j \frac{u_{ij}\cos(\theta_j^T)\sin(\theta_j^T)J{N_j^T}^t\cdot N_j^T}{2|N_j^T|^3}[A1]\\
&-\sum_j \frac{u_{ij}\sin^2(\theta_j^T)(2\eta-p_{t_{j+2}}-p_{t_{j+1}})}{2|N_j^T|^2}[A2]\\
&-\sum_j \frac{\theta_j^T u_{ij}J{N_j^T}^t\cdot N_j^T}{2|N_j^T|^3}[B]\\
&-\sum_j \frac{2u_{jj}({N_i^T}^t\cdot N_j^T)\cos(\theta_j^T)\sin(\theta_j^T)^3J{N_j^T}^t\cdot N_j^T}{4|N_j^T|^5}[C1]\\
&+\sum_j \frac{u_{jj}({N_i^T}^t\cdot N_j^T)\sin^4(\theta_j^T)(2\eta-p_{t_{j+2}}-p_{t_{j+1}})}{2|N_j^T|^4}[C2]\\
&+\sum_j \frac{\bar{e_1}(\theta_j^T)u_{jj}(J{N_j^T}^t\cdot N_i^T+J{N_i^T}^t\cdot N_j^T)}{4|N_j^T|^3}[D]\\
&-\sum_j \frac{3\bar{e_1}(\theta_j^T)u_{jj}({N_i^T}^t\cdot N_j^T)J{N_j^T}^t\cdot N_j^T}{4|N_j^T|^5}[E]\\
&-\sum_j \frac{\sin^2(\theta_j^T)\cos^2(\theta_j^T)({N_i^T}^t\cdot N_j^T)J{N_j^T}^t\cdot N_j^T}{|N_j^T|^4}[F1]\\
&+\sum_j \frac{\sin^3(\theta_j^T)\cos(\theta_j^T)({N_i^T}^t\cdot N_j^T)(2\eta-p_{t_{j+2}}-p_{t_{j+1}})}{|N_j^T|^3}[F2]\\
&-\sum_j \frac{\sin^2(\theta_j^T)(J{N_j^T}^t\cdot N_i^T+J{N_i^T}^t\cdot N_j^T)}{2|N_j^T|^2}[G]\\
&+\sum_j \frac{\sin^2(\theta_j^T)({N_i^T}^t\cdot N_j^T)J{N_j^T}^t\cdot N_j^T}{|N_j^T|^4}[H]
\end{aligned}
$$

We re-arrange the equations together, as

$$
\begin{aligned}
-2|T|\overrightarrow{\bigtriangledown} dw_i^T =& + \sum_j \frac{\bar{e_1}(\theta_j^T) u_{ij} J{N_j^T}^t \cdot N_j^T}{2|N_j^T|^3} [A1+B] \\
& - \sum_j \frac{u_{ij}\sin^2(\theta_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{2|N_j^T|^2} [B] \\
& - \sum_j \frac{u_{jj}\bar{e_4}(\theta_j^T)({N_i^T}^t \cdot N_j^T) J{N_j^T}^t \cdot N_j^T}{4|N_j^T|^5} [C1+E] \\
& + \sum_j \frac{u_{jj}({N_i^T}^t \cdot N_j^T)\sin^4(\theta_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{2|N_j^T|^4} [C2] \\
& + \sum_j \frac{\bar{e_1}(\theta_j^T) u_{jj}(J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{4|N_j^T|^3} [D] \\
& + \sum_j \frac{\sin^4(\theta_j^T)({N_i^T}^t \cdot N_j^T) J{N_j^T}^t \cdot N_j^T}{|N_j^T|^4} [F1+H] \\
& + \sum_j \frac{\sin^3(\theta_j^T)\cos(\theta_j^T)({N_i^T}^t \cdot N_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{|N_j^T|^3} [F2] \\
& - \sum_j \frac{\sin^2(\theta_j^T)(J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{2|N_j^T|^2} [G]
\end{aligned}
$$

Finally,

$$
\begin{aligned}
-2|T|\overrightarrow{\bigtriangledown} dw_i^T =& + \sum_j \frac{e_1(\theta_j^T)((p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}})) J{N_j^T}^t \cdot N_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
& - \sum_j \frac{((p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}}))(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} \\
& - \sum_j \frac{e_7(\theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2 ({N_i^T}^t \cdot N_j^T) J{N_j^T}^t \cdot N_j^T}{4(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^5} \\
& + \sum_j \frac{|p_{t_{j+2}} - p_{t_{j+1}}|^2 ({N_i^T}^t \cdot N_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^4} \\
& + \sum_j \frac{e_1(\theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2 (J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{4(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
& + \sum_j \frac{({N_i^T}^t \cdot N_j^T) J{N_j^T}^t \cdot N_j^T}{(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^4} \\
& + \sum_j \frac{\cos(\theta_j^T)({N_i^T}^t \cdot N_j^T)(2\eta - p_{t_{j+2}} - p_{t_{j+1}})}{(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\
& - \sum_j \frac{(J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2}
\end{aligned}
$$

where $e_7(x) = \frac{2\cos(x)\sin(x)^3 + 3(\sin(x)\cos(x) - x)}{\sin(x)^5}$.

9 Details of Taylor expansion of $w_{e_i}^E(\eta+\epsilon n_E)$ in the 2D case

Expression of $\frac{\partial(w_{e_i}^E(\eta+\epsilon n_E))}{\partial\epsilon}_{|\epsilon\to 0}$ To obtain a closed-form expression of $\frac{\partial(w_{e_i}^E(\eta+\epsilon n_E))}{\partial\epsilon}_{|\epsilon\to 0}$, we derive the Taylor expansion of $w_{e_i}^E(\eta+\epsilon n_E)$ with respect to ϵ, using Eq. 7.

$$w_{e_i}^E(\eta+\epsilon n_E)=\frac{m^E(\eta+\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)}{(p_{e_i}-\eta-\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)} \tag{47}$$

The denominator of the latter fraction is equal to:

$$\begin{aligned}(p_{e_i}-\eta-\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)=&(p_{e_i}-\eta-\epsilon n_E)^t\cdot R_{\frac{\pi}{2}}\cdot(p_{e_{i+1}}-\eta-\epsilon n_E)\cdot(-1)^i\\ =&(p_{e_i}-\eta)^t\cdot R_{\frac{\pi}{2}}\cdot(p_{e_{i+1}}-\eta)\cdot(-1)^i-\epsilon n_E^t\cdot R_{\frac{\pi}{2}}\cdot(p_{e_{i+1}}-\eta)\cdot(-1)^i\\ &-\epsilon(p_{e_i}-\eta)^t\cdot R_{\frac{\pi}{2}}\cdot n_E\cdot(-1)^i+\epsilon^2 n_E^t R_{\frac{\pi}{2}} n_E\cdot(-1)^i\\ =&\epsilon n_E^t\cdot R_{\frac{\pi}{2}}\cdot(p_{e_i}-p_{e_{i+1}})\cdot(-1)^i\\ =&-\epsilon|E|\end{aligned} \tag{48}$$

The numerator is equal to:

$$m^E(\eta+\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)=\sum_j\frac{N_j^E(\eta+\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)}{|N_j^E(\eta+\epsilon n_E)|} \tag{49}$$

We develop $N_j^E(\eta+\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)$ to the second order:

$$\begin{aligned}N_j^E(\eta+\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)=&(N_j^E(\eta)+\epsilon(-1)^{j+1}R_{\frac{\pi}{2}}\cdot n_E)^t\cdot(N_{i+1}^E(\eta)+\epsilon(-1)^i R_{\frac{\pi}{2}}\cdot n_E)\\ =&N_j^E(\eta)^t\cdot N_{i+1}(\eta)+\epsilon(-1)^{j+1}n_E^t\cdot N_{i+1}^E(\eta)\\ &+\epsilon(-1)^i N_j^E(\eta)^t\cdot R_{\frac{\pi}{2}}\cdot n_E+\epsilon^2(-1)^{i+j+1}(R_{\frac{\pi}{2}}\cdot n_E)^t\cdot(R_{\frac{\pi}{2}}\cdot n_E)\\ =&N_j^E(\eta)^t\cdot N_{i+1}(\eta)+\epsilon^2(-1)^{i+j+1}\end{aligned} \tag{50}$$

We develop $\frac{1}{|N_j^E(\eta+\epsilon n_E)|}$ to the second order:

$$\begin{aligned}\frac{1}{|N_j^E(\eta+\epsilon n_E)|}=&\frac{1}{|N_j^E|}-\epsilon\frac{(JN_j^{E^t}\cdot N_j^E)^t\cdot n_E}{|N_j^E|^3}\\ &-\frac{\epsilon^2}{2}\frac{n_E^t\cdot(JN_j^{E^t}\cdot JN_j^E)\cdot n_E}{|N_j^E|^3}\\ &+3\frac{\epsilon^2}{2}\frac{n_E^t\cdot(JN_j^{E^t}\cdot N_j^E\cdot N_j^{E^t}\cdot JN_j^E)\cdot n_E}{|N_j^E|^5}\\ =&\frac{1}{|N_j^E|}-\epsilon^2\frac{1}{2|N_j^E|^3}\end{aligned} \tag{51}$$

We obtain the Taylor expansion of $m^E(\eta+\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)$ as:

$$\begin{aligned}m^E(\eta+\epsilon n_E)^t\cdot N_{i+1}^E(\eta+\epsilon n_E)=&\sum_j(N_j^E(\eta)^t\cdot N_{i+1}(\eta)+\epsilon^2(-1)^{i+j+1})(\frac{1}{|N_j^E|}-\epsilon^2\frac{1}{2|N_j^E|^3})+o(\epsilon^2)\\ =&m^E(\eta)^t\cdot N_{i+1}^E(\eta)-\epsilon^2\sum_j\frac{N_j^E(\eta)^t\cdot N_{i+1}^E(\eta)}{2|N_j^E(\eta)|^3}+\frac{(-1)^{i+j}}{|N_j^E(\eta)|}+o(\epsilon^2)\\ =&-\epsilon^2\sum_j\frac{N_j^E(\eta)^t\cdot N_{i+1}^E(\eta)}{2|N_j^E(\eta)|^3}+\frac{(-1)^{i+j}}{|N_j^E(\eta)|}+o(\epsilon^2)\end{aligned} \tag{52}$$

We obtain that, for all $\eta\in(p_{e_0}p_{e_1}),\notin[p_{e_0}p_{e_1}]$:

$$\overrightarrow{\nabla}w_{t_i}^E(\eta)=(\sum_j\frac{N_j^E(\eta)^t\cdot N_{i+1}^E(\eta)}{2|E||N_j^E(\eta)|^3}+\frac{(-1)^{i+j}}{|E||N_j^E(\eta)|})n_E \tag{53}$$

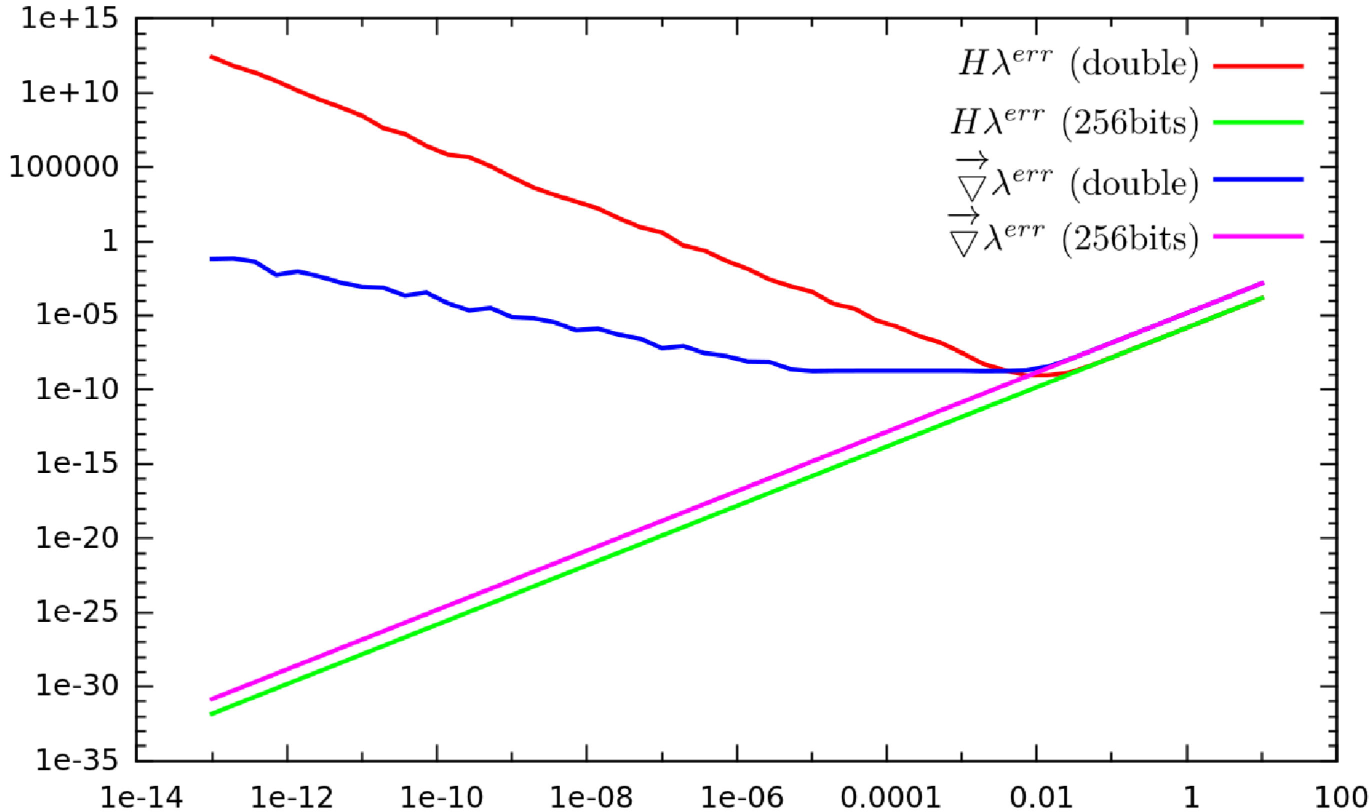


Figure 2: **Comparison with Finite Differences:** the domain are the same as described previously in Fig. 5 of the paper, and the evaluations are performed on point 0. x axis: size of the stencil for Finite Differences. y axis: difference between Finite Differences approximations of the derivatives and our formulae. Axes of the plots are in logarithmic scale. The functions that are plotted are $\vec{\bigtriangledown}\lambda^{err}(r) = \sqrt{\sum_i ||\vec{\bigtriangledown}\lambda_i - \vec{\bigtriangledown}\lambda_i{}^{FD(r)}||^2}$ and $H\lambda^{err}(r) = \sqrt{\sum_i ||H\lambda_i - H\lambda_i{}^{FD(r)}||^2}$.

10 Comparison with Finite Difference schemes in 3D

In this section we use Finite Differences schemes to derive the gradient and the Hessian of the MVC, to compare with the expressions we obtained.

A conventional scheme for approximations of first and second order derivatives at point (x, y, z) is the following:

$f_x \simeq \frac{f(x+h,y,z)-f(x-h,y,z)}{2h}$

$f_{xx} \simeq \frac{f(x+h,y,z)-2f(x,y,z)+f(x-h,y,z)}{h^2}$

$f_{xy} \simeq \frac{f(x+h,y+h,z)-f(x+h,y-h,z)-f(x-h,y+h,z)+f(x-h,y-h,z)}{4h^2}$

...

This scheme requires 19 evaluations of the function in total. Results of convergence of Finite Differences (FD) of the Mean Value Coordinates derivatives using this scheme are presented on an example in Fig. 2, using double precision and 256 bits precision (using **mpfrc++**, which is a c++ wrapper of the *GNU multiple precision floating point library (mpfr)*). The domain is the same as described previously in Fig. 5 of the paper, and the plots correspond here to the evaluation made in *point* 0. The error functions that are plotted are $\vec{\bigtriangledown}\lambda^{err}(r) = \sqrt{\sum_i ||\vec{\bigtriangledown}\lambda_i - \vec{\bigtriangledown}\lambda_i{}^{FD(r)}||^2}$ and $H\lambda^{err}(r) = \sqrt{\sum_i ||H\lambda_i - H\lambda_i{}^{FD(r)}||^2}$. Note, that these plots are representative of all the experiments we made (i. e. with other cages, at other locations, etc.).

These results validate empirically our formulae, as the Finite Differences scheme converges to our formulae when the size of the stencil tends to 0 (Fig. 2, using 256 bits precision). It also indicates that Finite Differences schemes are not suited to evaluate MVC derivatives in real life applications, as these schemes diverge near 0 when using double precision only (Fig. 2 blue and red curves). Note, that this behavior is not typical of Mean Value Coordinates, but rather of finite differences schemes. The choice of the size of the stencil is a typical difficulty in finite difference schemes. Choosing a size which is too small may introduce large rounding errors [Flannery et al.(1992)Flannery, Press, Teukolsky, and Vetterling, Squire and Trapp(1998)]. Finding the smallest size which minimizes rounding error is both machine dependent and application dependent (in our case $\simeq 0.01$ on the example of Fig. 2). Moreover, its has been shown that all finite difference formulae are ill-conditioned [Fornberg(1981)] and suffer from this drawback. We used different schemes to approximate the derivatives

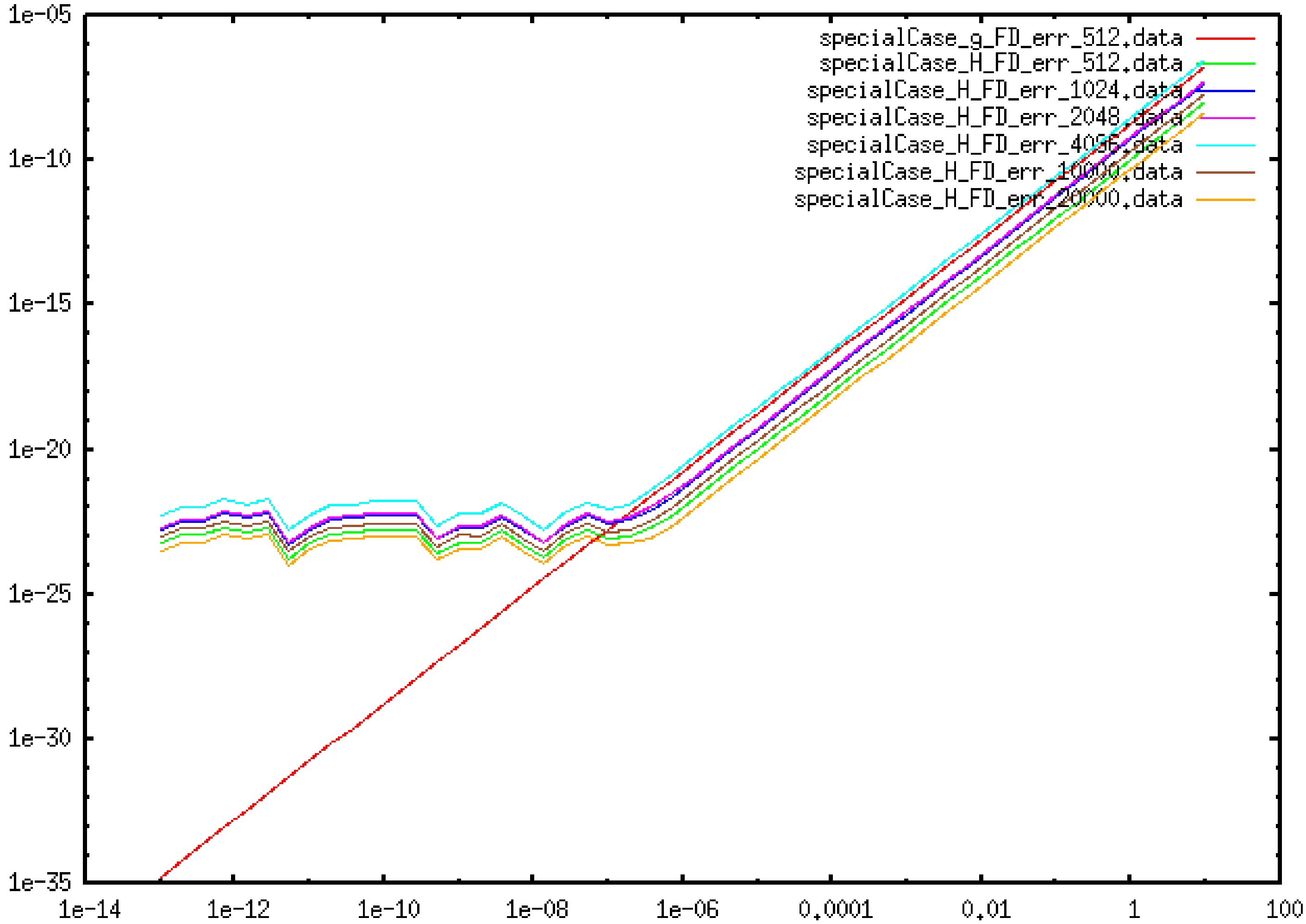


Figure 3: **Comparison with Finite Differences:** the domain are the same as described previously in Fig. 5 of the paper. x axis: size of the stencil for Finite Differences. y axis: difference between Finite Differences approximations of the derivatives and our formulae. Axes of the plots are in logarithmic scale.

using Finite Differences methods (9 points evaluation + linear system inversion, 19 points evaluation on a $3 \times 3 \times 3$-stencil, tricubic interpolation on a $4 \times 4 \times 4$-stencil), and that they all diverge in the same manner when using double precision.

The error curves are also similar when looking at the deviation of the gradients and Hessians of the **function itself** that is interpolated (e. g. a deformation function), instead of the gradients and Hessians of the **weights** themselves.

Evaluation against Finite Differences in the degenerate cases

We also performed a validation against finite differences schemes to validate our formulae given in the particular case of the point η lying on the support of the triangles. Fig. 3 presents the experiment.

The functions that are plotted are the same as before, except that we compute the errors not for all MVC weights, but only for those regarding the triangle on which the point lies.

We can see that the gradient seems to be validated by the experiment, as the error curve goes to 0 when approximating the gradient with Finite Differences using a stencil's size that tends to 0. For the Hessian however, we can observe fluctuations of the error around a value of 10^{-23}.

We recall, that the x-axis is the stencil size h, and that the y-axis is the difference between our formula and the finite difference approximation using the stencil size h.

The finite differences scheme does not seem to converge in this experiment (otherwise, there would be stabilization of the difference around a value, eventually 0 if it validates our formula, but there is no stabilization of the error whatsoever, even when using a precision of 20,000 bits for the computation). One reason might be that using the general formula for the computation of the weights is unstable on the support plane of the triangle (we recall that it corresponds to a division of 0 by 0, and that [Ju et al.(2005)Ju, Schaefer, and Warren] handles the case by setting the value of the weights to 0, as it corresponds to the limit of the weights at these positions). In our case, we cannot use the same trick and set the weights to 0 when the points are near the support of the triangles based on a simple threshold ϵ, since, as the stencil size h tends to 0, all evaluation points are at distance smaller than ϵ, and the finite differences computation would simply give null derivatives.

Note however, that the fluctuations of the error is of the order of 10^{-18} times the actual norm of the Hessian (which gives a high signal to noise ratio of 10^{18}). In any way, the finite difference scheme does not converge to a value different from the one given by our formula. Note also, that the value of the Hessian is obtained directly from the differentiation of the gradient that is validated by the finite difference approximation.

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