The complexity of carry propagation for successor functions

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Adding machine

The Pascaline (1642)

featured the first carry propagation mechanism

Carry propagation

1	1	0	0	1	0	1	1	203
	1	1	0	0	1	1	0	102

Carry propagation

	1	1	0	0	1	0	1	1		203
+		1	1	0	0	1	1	0		102

Carry propagation

$$
\begin{array}{r r}
 & 1\,1\,0\,0\,1\,0\,1\,1 \\
+ & 1\,1\,0\,0\,1\,1\,0 \\
\hline
 & 1
\end{array}
$$

203

102

Carry propagation

	1	1	0	0	1	0	1	1	203
+		1	1	0	0	1	1	0	102
							0	1	

Carry propagation

	1	1	0	0	1	0	1	1		203
+		1	1	0	0	1	1	0		102
						0	0	1		

Carry propagation

	1	1	0	0	1	0	1	1		203
+		1	1	0	0	1	1	0		102
					0	0	0	1		

Carry propagation

	1	1	0	0	1	0	1	1	203
+		1	1	0	0	1	1	0	102
				1	0	0	0	1	

Carry propagation

	1	1	0	0	1	0	1	1	203
+		1	1	0	0	1	1	0	102
			1	1	0	0	0	1	

Carry propagation

	1 1 0 0 1 0 1 1	203
+	1 1 0 0 1 1 0	102
	0 1 1 0 0 0 1	

Carry propagation

	1	1	0	0	1	0	1	1	203
+		1	1	0	0	1	1	0	102
	0	0	1	1	0	0	0	1	

Carry propagation

		1	1	0	0	1	0	1	1	203
+			1	1	0	0	1	1	0	102
	1	0	0	1	1	0	0	0	1	305

Carry propagation

		1	1	0	0	1	0	1	1	203
+			1	1	0	0	1	1	0	102
	1	0	0	1	1	0	0	0	1	305

Carry propagation prevents addition to be parallelable

Carry propagation

		1	1	0	0	1	0	1	1	203
+			1	1	0	0	1	1	0	102
	1	0	0	1	1	0	0	0	1	305

Theorem (von Neumann et al. 63, Knuth 78, Pippenger 02)

Average carry propagation length for addition of two uniformly distributed n-digit binary numbers =

$$\log_2(n) + \mathrm{O}(1)$$

Carry propagation for successor function in base 2

1 1 0 0 1 0 1 1   203

Carry propagation for successor function in base 2

$$
\begin{array}{r}
1\,1\,0\,0\,1\,0\,1\,1 \\
+ \qquad\qquad\qquad\quad 1 \\
\hline
\end{array}
\qquad 203
$$

Carry propagation for successor function in base 2

$$\begin{array}{rcccccccc} & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ + & & & & & & & & 1 \\ \hline & & & & & & & & 0 \end{array} \qquad 203$$

Carry propagation for successor function in base 2

$$\begin{array}{r} 1\,1\,0\,0\,1\,0\,1\,1 \\ +\qquad\qquad\qquad 1 \\ \hline 0\,0 \end{array}$$

203

Carry propagation for successor function in base 2

$$
\begin{array}{rr}
 & 1\,1\,0\,0\,1\,0\,1\,1 \\
+ & 1 \\
\hline
 & 1\,0\,0
\end{array}
$$

203

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccc}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & & 1 \\
\hline
 & & & & & 1 & 1 & 0 & 0
\end{array}
$$

203

Carry propagation for successor function in base 2

$$\begin{array}{rcccccccc} & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ + & & & & & & & & 1 \\ \hline & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}$$

203

204

Carry propagation for successor function in base 2

	1	1	0	0	1	0	1	1	203
+								1	
	1	1	0	0	1	1	0	0	204

$$\mathrm{cp}_2(203) = 3$$

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccccc}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & & & & & & & 1 & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & & 204
\end{array}
$$

$$\mathrm{cp}_2(203) = 3$$

Amortized carry propagation (in base 2)

$$\mathrm{CP}_2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_2(i)$$

Carry propagation for successor function in base 2

$$\begin{array}{rcccccccc} & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ + & & & & & & & & 1 \\ \hline & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}$$

203

204

$$\mathrm{cp}_2(203) = 3$$

Amortized carry propagation (in base 2)

$$\mathrm{CP}_2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_2(i)$$

if it exists!

Carry propagation for successor function in base 2

0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

Carry propagation for successor function in base 2

·	0
1	1
1 0	2
1 1	3
1 0 0	4
1 0 1	5
1 1 0	6
1 1 1	7
1 0 0 0	8
1 0 0 1	9
1 0 1 0	10
1 0 1 1	11
1 1 0 0	12
1 1 0 1	13
1 1 1 0	14
1 1 1 1	15
1 0 0 0 0	16
1 0 0 0 1	17

Carry propagation for successor function in base 2

·	0	1
1	1	
1 0	2	
1 1	3	
1 0 0	4	
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	
1 1	3	
1 0 0	4	
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	
1 0 0	4	
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	2
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	2
1 1 0	6	1
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	2
1 1 0	6	1
1 1 1	7	4
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	2
1 1 0	6	1
1 1 1	7	4
1 0 0 0	8	1
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	2
1 1 0	6	1
1 1 1	7	4
1 0 0 0	8	1
1 0 0 1	9	2
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	2
1 1 0	6	1
1 1 1	7	4
1 0 0 0	8	1
1 0 0 1	9	2
1 0 1 0	10	1
1 0 1 1	11	3
1 1 0 0	12	1
1 1 0 1	13	2
1 1 1 0	14	1
1 1 1 1	15	5
1 0 0 0 0	16	1
1 0 0 0 1	17	2

Carry propagation for successor function in base 2

·	0	1
1	1	1 1
1 0	2	1
1 1	3	1 1 1
1 0 0	4	1
1 0 1	5	1 1
1 1 0	6	1
1 1 1	7	1 1 1 1
1 0 0 0	8	1
1 0 0 1	9	1 1
1 0 1 0	10	1
1 0 1 1	11	1 1 1
1 1 0 0	12	1
1 1 0 1	13	1 1
1 1 1 0	14	1
1 1 1 1	15	1 1 1 1 1
1 0 0 0 0	16	1
1 0 0 0 1	17	1 1

Carry propagation for successor function in base 2

$$\mathrm{CP}_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2-1} = 2$$

·	0	1
1	1	1 1
1 0	2	1
1 1	3	1 1 1
1 0 0	4	1
1 0 1	5	1 1
1 1 0	6	1
1 1 1	7	1 1 1 1
1 0 0 0	8	1
1 0 0 1	9	1 1
1 0 1 0	10	1
1 0 1 1	11	1 1 1
1 1 0 0	12	1
1 1 0 1	13	1 1
1 1 1 0	14	1
1 1 1 1	15	1 1 1 1 1
1 0 0 0 0	16	1
1 0 0 0 1	17	1 1

Carry propagation for successor function in base p

$$\mathrm{CP}_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2-1} = 2$$

$$\mathrm{CP}_p = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{p}{p-1}$$

·	0	1
1	1	1 1
1 0	2	1
1 1	3	1 1 1
1 0 0	4	1
1 0 1	5	1 1
1 1 0	6	1
1 1 1	7	1 1 1 1
1 0 0 0	8	1
1 0 0 1	9	1 1
1 0 1 0	10	1
1 0 1 1	11	1 1 1
1 1 0 0	12	1
1 1 0 1	13	1 1
1 1 1 0	14	1
1 1 1 1	15	1 1 1 1 1
1 0 0 0 0	16	1
1 0 0 0 1	17	1 1

Carry propagation for successor function in base Fibonacci

1 0 0 1 0 1 0 1

Carry propagation for successor function in base Fibonacci

55	34	21	13	8	5	3	2	1	
	1	0	0	1	0	1	0	1	46

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

$$\mathrm{cp}_F(46) = 6$$

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

$$\mathrm{cp}_F(46) = 6$$

Amortized carry propagation in base Fibonacci

$$\mathrm{CP}_F = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_F(i)$$

if it exists!

Carry propagation for successor function in base Fibonacci

·	0
1	1
1 0	2
1 0 0	3
1 0 1	4
1 0 0 0	5
1 0 0 1	6
1 0 1 0	7
1 0 0 0 0	8
1 0 0 0 1	9
1 0 0 1 0	10
1 0 1 0 0	11
1 0 1 0 1	12
1 0 0 0 0 0	13
1 0 0 0 0 1	14
1 0 0 0 1 0	15
1 0 0 1 0 0	16
1 0 0 1 0 1	17

Carry propagation for successor function in base Fibonacci

·	0	1
1	1	2
1 0	2	3
1 0 0	3	1
1 0 1	4	4
1 0 0 0	5	1
1 0 0 1	6	2
1 0 1 0	7	5
1 0 0 0 0	8	1
1 0 0 0 1	9	2
1 0 0 1 0	10	3
1 0 1 0 0	11	1
1 0 1 0 1	12	6
1 0 0 0 0 0	13	1
1 0 0 0 0 1	14	2
1 0 0 0 1 0	15	3
1 0 0 1 0 0	16	1
1 0 0 1 0 1	17	4

Carry propagation for successor function in base Fibonacci

CP_F ?

·	0	1
1	1	2
1 0	2	3
1 0 0	3	1
1 0 1	4	4
1 0 0 0	5	1
1 0 0 1	6	2
1 0 1 0	7	5
1 0 0 0 0	8	1
1 0 0 0 1	9	2
1 0 0 1 0	10	3
1 0 1 0 0	11	1
1 0 1 0 1	12	6
1 0 0 0 0 0	13	1
1 0 0 0 0 1	14	2
1 0 0 0 1 0	15	3
1 0 0 1 0 0	16	1
1 0 0 1 0 1	17	4

Carry propagation for successor function in base Fibonacci

CP_F ?

$$\mathrm{CP}_F = \frac{\varphi}{\varphi - 1} \quad ?$$

·	0	1
1	1	2
1 0	2	3
1 0 0	3	1
1 0 1	4	4
1 0 0 0	5	1
1 0 0 1	6	2
1 0 1 0	7	5
1 0 0 0 0	8	1
1 0 0 0 1	9	2
1 0 0 1 0	10	3
1 0 1 0 0	11	1
1 0 1 0 1	12	6
1 0 0 0 0 0	13	1
1 0 0 0 0 1	14	2
1 0 0 0 1 0	15	3
1 0 0 1 0 0	16	1
1 0 0 1 0 1	17	4

Carry propagation for successor function in base $3/2$

2 1 0 1 1 2 1

Carry propagation for successor function in base $3/2$

$\frac{1}{2}(\frac{3}{2})^7$	$\frac{1}{2}(\frac{3}{2})^6$	$\frac{1}{2}(\frac{3}{2})^5$	$\frac{1}{2}(\frac{3}{2})^4$	$\frac{1}{2}(\frac{3}{2})^3$	$\frac{1}{2}(\frac{3}{2})^2$	$\frac{1}{2}\frac{3}{2}$	$\frac{1}{2}$
	2	1	0	1	1	2	1

Carry propagation for successor function in base $3/2$

$\frac{1}{2}(\frac{3}{2})^7$	$\frac{1}{2}(\frac{3}{2})^6$	$\frac{1}{2}(\frac{3}{2})^5$	$\frac{1}{2}(\frac{3}{2})^4$	$\frac{1}{2}(\frac{3}{2})^3$	$\frac{1}{2}(\frac{3}{2})^2$	$\frac{1}{2}\frac{3}{2}$	$\frac{1}{2}$	
	2	1	0	1	1	2	1	20

A quick look at the base $\frac{3}{2}$ numeration system

Integer representations in base 3: the Euclidean approach

$$V = \left\{ v_i = (3)^i \;\middle|\; i \in \mathbb{N} \right\} \qquad \text{together with} \qquad A_3 = \{0, 1, 2\}$$

Division algorithm $\quad N \in \mathbb{N}$

$$N_0 = N$$

$$N_0 = 3\,N_1 + a_0 \qquad a_0 \in A$$

$$N_1 = 3\,N_2 + a_1 \qquad a_1 \in A$$

$$\dots$$

$$N = \sum_0^k a_i\,3^i \qquad \langle N \rangle_3 = a_k\,a_{k-1}\dots a_1\,a_0$$

A quick look at the base $\frac{3}{2}$ numeration system

Integer representations in base 3: the Euclidean approach

$$V = \left\{ v_i = (3)^i \;\middle|\; i \in \mathbb{N} \right\} \qquad \text{together with} \qquad A_3 = \{0, 1, 2\}$$

Division algorithm $\qquad 17 \in \mathbb{N}$

$$\begin{aligned} N_0 &= 17 \\ 17 = N_0 &= 3 \cdot 5 + 2 & \qquad a_0 = 2 \in A \\ 5 = N_1 &= 3 \cdot 1 + 2 & \qquad a_1 = 2 \in A \\ 1 = N_2 &= 3 \cdot 0 + 1 & \qquad a_2 = 1 \in A \end{aligned}$$

$$17 = ((1) \cdot 3 + 2) \cdot 3 + 2 \qquad\qquad \langle 17 \rangle_3 = 1\,2\,2$$

A quick look at the base $\frac{3}{2}$ numeration system

Integer representations in base $\frac{3}{2}$: the Euclidean approach

$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\}$ together with $A_3 = \{0, 1, 2\}$

Modified division algorithm $N \in \mathbb{N}$

$$\begin{aligned} N_0 &= N \\ 2\,N_0 &= 3\,N_1 + a_0 & a_0 \in A \\ 2\,N_1 &= 3\,N_2 + a_1 & a_1 \in A \\ &\dots \end{aligned}$$

$$N = \sum_0^k a_i \frac{1}{2} \left(\frac{3}{2}\right)^i \qquad \langle N \rangle_{\frac{3}{2}} = a_k\, a_{k-1} \dots a_1\, a_0$$

A quick look at the base $\frac{3}{2}$ numeration system

Integer representations in base $\frac{3}{2}$: the Euclidean approach

$$U = \left\{ u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \;\middle|\; i \in \mathbb{N} \right\} \quad \text{together with} \quad A_3 = \{0, 1, 2\}$$

Modified division algorithm $\quad 5 \in \mathbb{N}$

$$\begin{aligned}
N_0 &= 5 & & \\
2\,N_0 &= 2 \cdot 5 = 3 \cdot 3 + 1 & & 1 \in A \\
2\,N_1 &= 2 \cdot 3 = 3 \cdot 2 + 0 & & 0 \in A \\
2\,N_2 &= 2 \cdot 2 = 3 \cdot 1 + 1 & & 1 \in A \\
2\,N_3 &= 2 \cdot 1 = 3 \cdot 0 + 2 & & 2 \in A
\end{aligned}$$

$$5 = \frac{1}{2}\left[\left(\left((2) \cdot \frac{3}{2} + 1 \right) \cdot \frac{3}{2} + 0 \right) \cdot \frac{3}{2} + 1 \right] \qquad \langle 5 \rangle_{\frac{3}{2}} = 2\,1\,0\,1$$

A quick look at the base $\frac{3}{2}$ numeration system

Theorem (Akiyama, Frougny, S. 08)

Every N *in* $\mathbb{N}$ *has an integer representation in the* $\frac{3}{2}$*-system.*

It is the unique finite $\frac{3}{2}$*-representation of* N *.*

A quick look at the base $\frac{3}{2}$ numeration system

Theorem (Akiyama, Frougny, S. 08)

Every N in $\mathbb{N}$ has an integer representation in the $\frac{3}{2}$-system.

It is the unique finite $\frac{3}{2}$-representation of N.

$$L_{\frac{3}{2}} = \left\{ \langle N \rangle_{\frac{3}{2}} \,\middle|\, N \in \mathbb{N} \right\} = ????$$

Some information in works
by Akiyama, Marsault, and S. (13–17)

Carry propagation for successor function in base $3/2$

	$\frac{1}{2}(\frac{3}{2})^7$	$\frac{1}{2}(\frac{3}{2})^6$	$\frac{1}{2}(\frac{3}{2})^5$	$\frac{1}{2}(\frac{3}{2})^4$	$\frac{1}{2}(\frac{3}{2})^3$	$\frac{1}{2}(\frac{3}{2})^2$	$\frac{1}{2}\frac{3}{2}$	$\frac{1}{2}$	
		2	1	0	1	1	2	1	20
+								1	

Carry propagation for successor function in base $3/2$

	$\frac{1}{2}(\frac{3}{2})^7$	$\frac{1}{2}(\frac{3}{2})^6$	$\frac{1}{2}(\frac{3}{2})^5$	$\frac{1}{2}(\frac{3}{2})^4$	$\frac{1}{2}(\frac{3}{2})^3$	$\frac{1}{2}(\frac{3}{2})^2$	$\frac{1}{2}\frac{3}{2}$	$\frac{1}{2}$	
		2	1	0	1	1	2	1	20
+								1	
		2	1	2	0	0	1	0	21

Carry propagation for successor function in base $3/2$

	$\frac{1}{2}(\frac{3}{2})^7$	$\frac{1}{2}(\frac{3}{2})^6$	$\frac{1}{2}(\frac{3}{2})^5$	$\frac{1}{2}(\frac{3}{2})^4$	$\frac{1}{2}(\frac{3}{2})^3$	$\frac{1}{2}(\frac{3}{2})^2$	$\frac{1}{2}\frac{3}{2}$	$\frac{1}{2}$	
		2	1	0	1	1	2	1	20
+								1	
		2	1	2	0	0	1	0	21

Carry propagation for successor function in base $3/2$

	$\frac{1}{2}(\frac{3}{2})^7$	$\frac{1}{2}(\frac{3}{2})^6$	$\frac{1}{2}(\frac{3}{2})^5$	$\frac{1}{2}(\frac{3}{2})^4$	$\frac{1}{2}(\frac{3}{2})^3$	$\frac{1}{2}(\frac{3}{2})^2$	$\frac{1}{2}\frac{3}{2}$	$\frac{1}{2}$	
		2	1	0	1	1	2	1	20
+								1	
		2	1	2	0	0	1	0	21

$$\mathrm{cp}_{\frac{3}{2}}(20) = 5$$

Carry propagation for successor function in base $3/2$

	$\frac{1}{2}(\frac{3}{2})^7$	$\frac{1}{2}(\frac{3}{2})^6$	$\frac{1}{2}(\frac{3}{2})^5$	$\frac{1}{2}(\frac{3}{2})^4$	$\frac{1}{2}(\frac{3}{2})^3$	$\frac{1}{2}(\frac{3}{2})^2$	$\frac{1}{2}\frac{3}{2}$	$\frac{1}{2}$	
		2	1	0	1	1	2	1	20
+								1	
		2	1	2	0	0	1	0	21

$$\mathrm{cp}_{\frac{3}{2}}(20) = 5$$

Amortized carry propagation in base $3/2$

$$\mathrm{CP}_{\frac{3}{2}} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_{\frac{3}{2}}(i)$$

if it exists!

Carry propagation for successor function in base $3/2$

·	0
2	1
2 1	2
2 1 0	3
2 1 2	4
2 1 0 1	5
2 1 2 0	6
2 1 2 2	7
2 1 0 1 1	8
2 1 2 0 0	9
2 1 2 0 2	10
2 1 2 2 1	11
2 1 0 1 1 0	12
2 1 0 1 1 2	13
2 1 2 0 0 1	14
2 1 2 0 2 0	15
2 1 2 0 2 2	16
2 1 2 2 1 1	17

Carry propagation for successor function in base 3/2

·	0	1
2	1	2
2 1	2	3
2 1 0	3	1
2 1 2	4	4
2 1 0 1	5	2
2 1 2 0	6	1
2 1 2 2	7	5
2 1 0 1 1	8	3
2 1 2 0 0	9	1
2 1 2 0 2	10	2
2 1 2 2 1	11	6
2 1 0 1 1 0	12	1
2 1 0 1 1 2	13	4
2 1 2 0 0 1	14	2
2 1 2 0 2 0	15	1
2 1 2 0 2 2	16	3
2 1 2 2 1 1	17	7

Carry propagation for successor function in base $3/2$

$CP_{\frac{3}{2}}$?

·	0	1
2	1	2
2 1	2	3
2 1 0	3	1
2 1 2	4	4
2 1 0 1	5	2
2 1 2 0	6	1
2 1 2 2	7	5
2 1 0 1 1	8	3
2 1 2 0 0	9	1
2 1 2 0 2	10	2
2 1 2 2 1	11	6
2 1 0 1 1 0	12	1
2 1 0 1 1 2	13	4
2 1 2 0 0 1	14	2
2 1 2 0 2 0	15	1
2 1 2 0 2 2	16	3
2 1 2 2 1 1	17	7

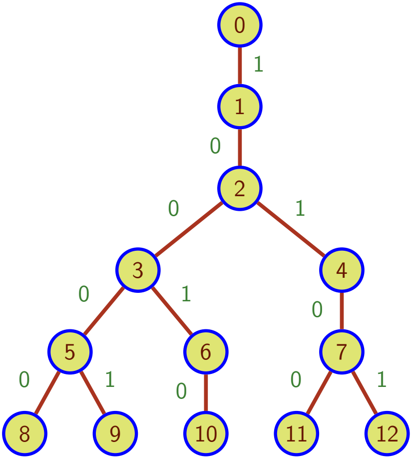
Carry propagation for successor function in base $3/2$

$\mathsf{CP}_{\frac{3}{2}}$?

$$\mathsf{CP}_{\frac{3}{2}} = \frac{\frac{3}{2}}{\frac{3}{2}-1} = 3$$?

·	0	1
2	1	2
2 1	2	3
2 1 0	3	1
2 1 2	4	4
2 1 0 1	5	2
2 1 2 0	6	1
2 1 2 2	7	5
2 1 0 1 1	8	3
2 1 2 0 0	9	1
2 1 2 0 2	10	2
2 1 2 2 1	11	6
2 1 0 1 1 0	12	1
2 1 0 1 1 2	13	4
2 1 2 0 0 1	14	2
2 1 2 0 2 0	15	1
2 1 2 0 2 2	16	3
2 1 2 2 1 1	17	7

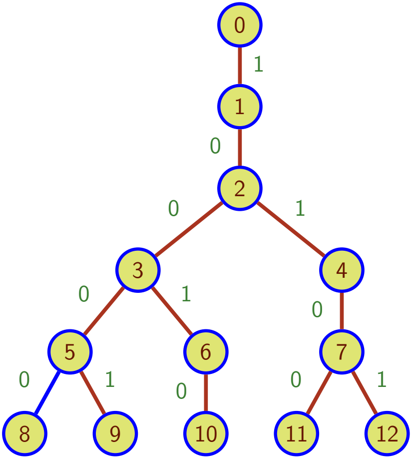
A primal observation



The Fibonacci tree

A primal observation

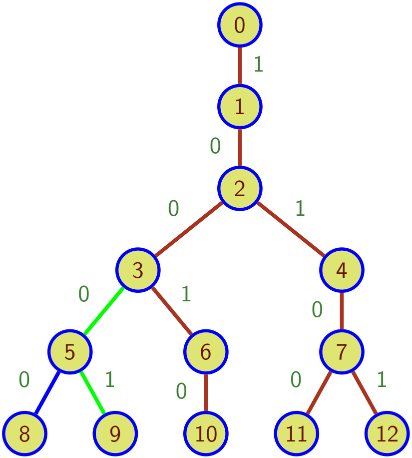
$\mathrm{cp}_F(8) = 1$



The Fibonacci tree

A primal observation

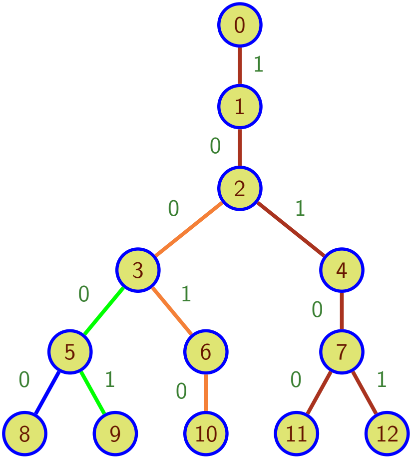
$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$



The Fibonacci tree

A primal observation

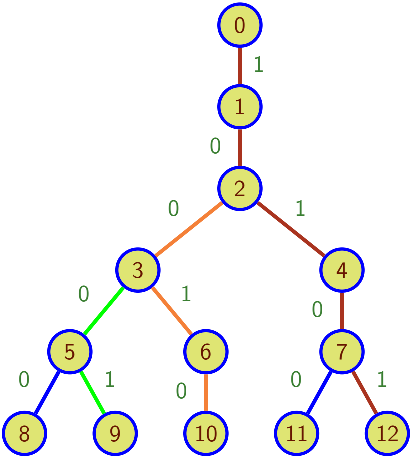
$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$
$\mathrm{cp}_F(10) = 3$



The Fibonacci tree

A primal observation

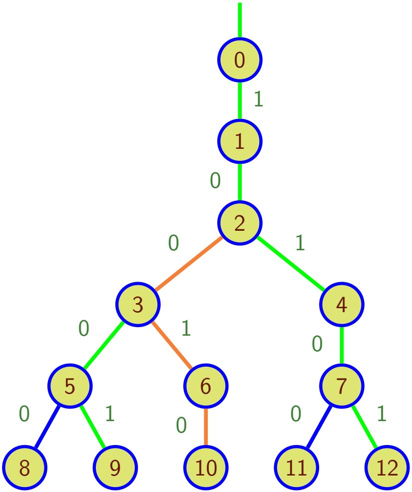
$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$
$\mathrm{cp}_F(10) = 3$
$\mathrm{cp}_F(11) = 1$



The Fibonacci tree

A primal observation

$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$
$\mathrm{cp}_F(10) = 3$
$\mathrm{cp}_F(11) = 1$
$\mathrm{cp}_F(12) = 6$



The Fibonacci tree

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$$\mathrm{cp}_F(8) = 1$$
$$\mathrm{cp}_F(9) = 2$$
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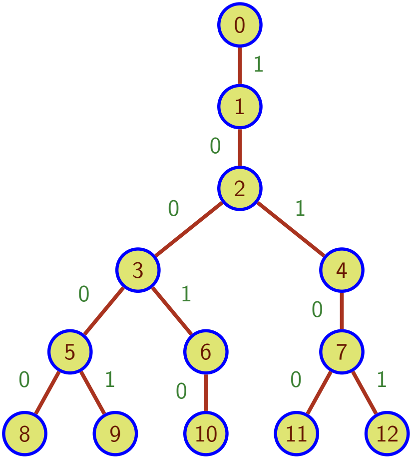
$$\sum_{i=8}^{i=12} \mathrm{cp}_F(i) = 13$$

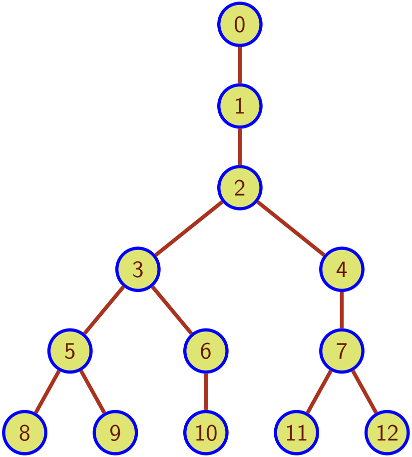


The Fibonacci tree

A primal observation

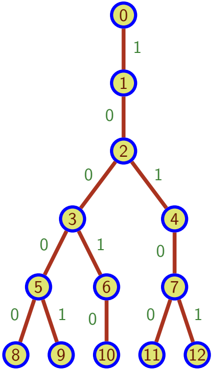
$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$
$\mathrm{cp}_F(10) = 3$
$\mathrm{cp}_F(11) = 1$
$\mathrm{cp}_F(12) = 6$

$$\sum_{i=8}^{i=12} \mathrm{cp}_F(i) = 13$$



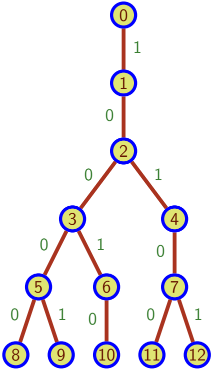
The Fibonacci tree

What we learn from the primal observation



What we learn from the primal observation

- A framework:
  the Abstract Numeration System model



What we learn from the primal observation

- A framework:
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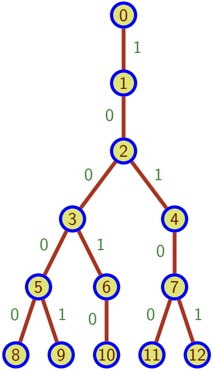
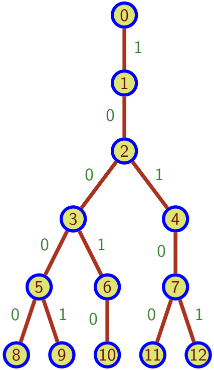
- A general working hypothesis:
  Prefix-closed Extendable Languages



What we learn from the primal observation

- A framework:
  the Abstract Numeration System model
- A general working hypothesis:
  Prefix-closed Extendable Languages
- An essential parameter:
  The local growth rate

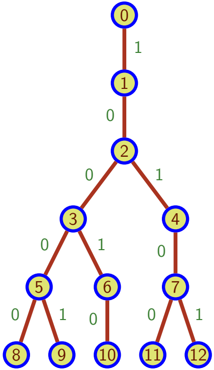


What we learn from the primal observation: the ANS model

- A framework:

  the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)



What we learn from the primal observation: the ANS model

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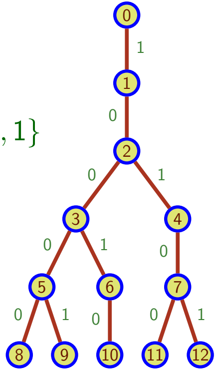
Definition (Lecomte & Rigo 2001)

- A finite *totally ordered* alphabet *e.g.* $A = \{0, 1\}$



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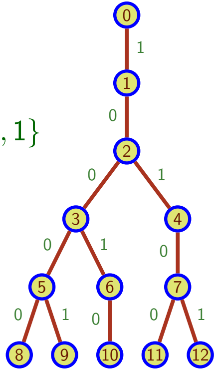

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i.e. ordered first *by length*, and then,
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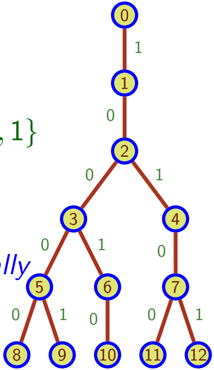

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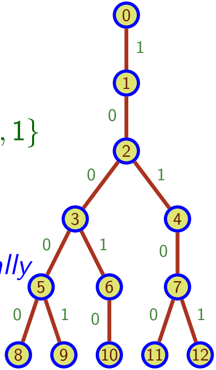

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e.g. $A^* = \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$

What we learn from the primal observation: the ANS model

- A framework:

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Definition (Lecomte & Rigo 2001)

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- $L \subseteq A^*$ any *language* over A^*

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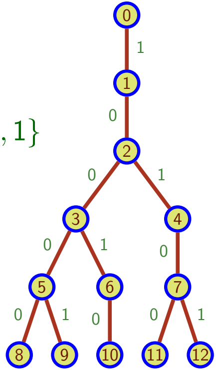

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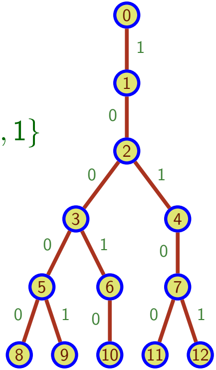

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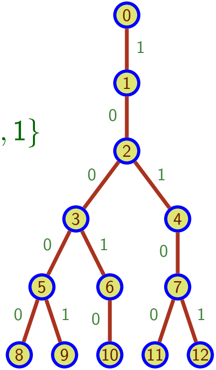
  $F = \varepsilon, 1, 10, 100, 101, 1000, \ldots$



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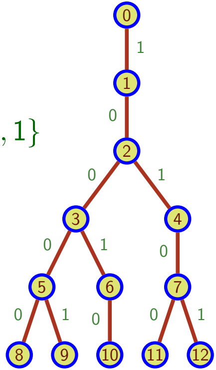
$\Rightarrow$ Natural integers are given *representations*
by means of words of L



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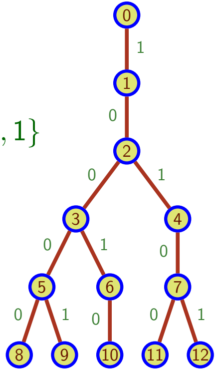

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What we learn from the primal observation: the ANS model

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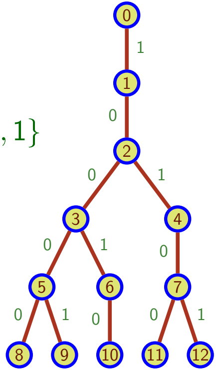

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i.e. $\langle n \rangle_L =$ $(n+1)$-th word of L in the radix ordering

e.g. $\langle 6 \rangle_F = 1001$

What we learn from the primal observation: the ANS model

- A framework:
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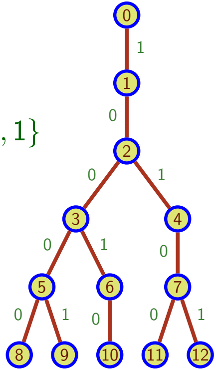

$L \subseteq A^*$ (together with the order on A) defines an ANS

What we learn from the primal observation: the ANS model

- A framework:
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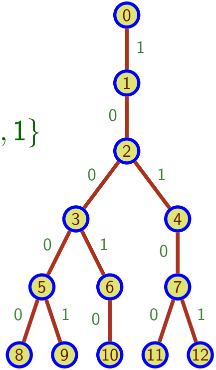

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$L \subseteq A^*$ (together with the order on A) defines an ANS

Fact: *All 'classical' numeration systems are ANS*

What we learn from the primal observation: the ANS model

All 'classical' numeration systems are ANS

0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

What we learn from the primal observation: the ANS model

All ‘classical’ numeration systems are ANS

$$L_2 = 1\,(0,1)^*$$

·	0
1	1
1 0	2
1 1	3
1 0 0	4
1 0 1	5
1 1 0	6
1 1 1	7
1 0 0 0	8
1 0 0 1	9
1 0 1 0	10
1 0 1 1	11
1 1 0 0	12
1 1 0 1	13
1 1 1 0	14
1 1 1 1	15
1 0 0 0 0	16

What we learn from the primal observation: the ANS model

All ‘classical’ numeration systems are ANS

$L_F = 1(0,1)^* \setminus (0,1)^* 1 1 (0,1)^*$

·	·	0
1	1	1
1 0	1 0	2
1 0 0	1 1	3
1 0 1	1 0 0	4
1 0 0 0	1 0 1	5
1 0 0 1	1 1 0	6
1 0 1 0	1 1 1	7
1 0 0 0 0	1 0 0 0	8
1 0 0 0 1	1 0 0 1	9
1 0 0 1 0	1 0 1 0	10
1 0 1 0 0	1 0 1 1	11
1 0 1 0 1	1 1 0 0	12
1 0 0 0 0 0	1 1 0 1	13
1 0 0 0 0 1	1 1 1 0	14
1 0 0 0 1 0	1 1 1 1	15
1 0 0 1 0 0	1 0 0 0 0	16

What we learn from the primal observation: the ANS model

All 'classical' numeration systems are ANS

$L_{\frac{3}{2}}$

.	.	.	0
2	1	1	1
2 1	1 0	1 0	2
2 1 0	1 0 0	1 1	3
2 1 2	1 0 1	1 0 0	4
2 1 0 1	1 0 0 0	1 0 1	5
2 1 2 0	1 0 0 1	1 1 0	6
2 1 2 2	1 0 1 0	1 1 1	7
2 1 0 1 1	1 0 0 0 0	1 0 0 0	8
2 1 2 0 0	1 0 0 0 1	1 0 0 1	9
2 1 2 0 2	1 0 0 1 0	1 0 1 0	10
2 1 2 2 1	1 0 1 0 0	1 0 1 1	11
2 1 0 1 1 0	1 0 1 0 1	1 1 0 0	12
2 1 0 1 1 2	1 0 0 0 0 0	1 1 0 1	13
2 1 2 0 0 1	1 0 0 0 0 1	1 1 1 0	14
2 1 2 0 2 0	1 0 0 0 1 0	1 1 1 1	15
2 1 2 0 2 2	1 0 0 1 0 0	1 0 0 0 0	16

What we learn from the primal observation: the ANS model

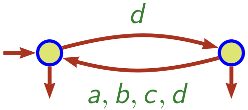
Any language can be seen as an ANS

What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language O

$\mathcal{O}$

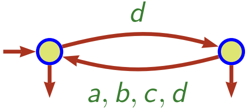


What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language O (for *Oscillating*)

$\mathcal{O}$



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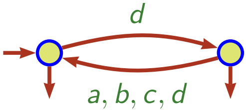

.
d
d a
d b
d c
d d
d a d
d b d
d c d
d d d
d a d a
d a d b
d a d c
d a d d
d b d a
d b d b
d b d c

What we learn from the primal observation: the ANS model

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Language O (for *Oscillating*)

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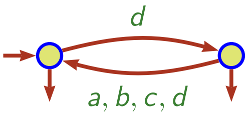

.	0
d	1
d a	2
d b	3
d c	4
d d	5
d a d	6
d b d	7
d c d	8
d d d	9
d a d a	10
d a d b	11
d a d c	12
d a d d	13
d b d a	14
d b d b	15
d b d c	16

What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language O (for *Oscillating*)

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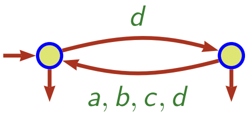


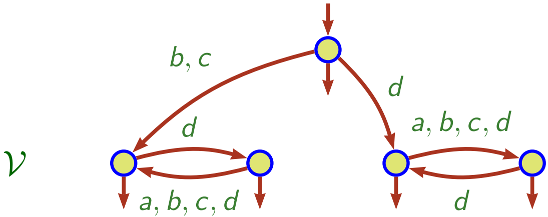
$$\langle 10 \rangle_O = d\,a\,d\,a$$

.	0
d	1
d a	2
d b	3
d c	4
d d	5
d a d	6
d b d	7
d c d	8
d d d	9
d a d a	10
d a d b	11
d a d c	12
d a d d	13
d b d a	14
d b d b	15
d b d c	16

What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

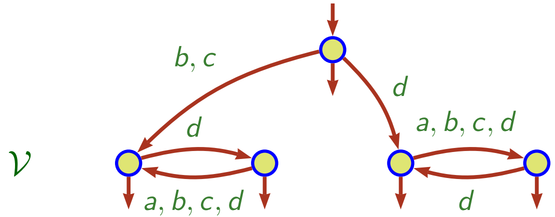
Language V



What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language V (for *Vibrating*)



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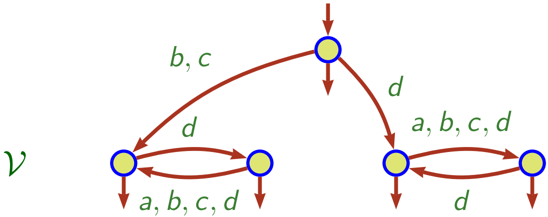

.
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d b
d c
d d
b d a
b d b
b d c
b d d
c d a
c d b
c d c

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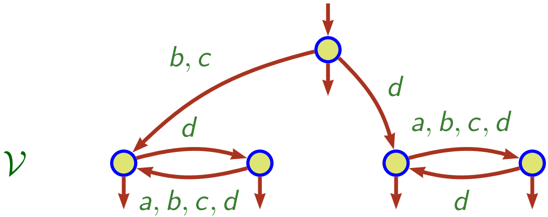


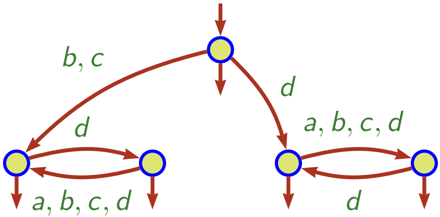
.	0
b	1
c	2
d	3
b d	4
c d	5
d a	6
d b	7
d c	8
d d	9
b d a	10
b d b	11
b d c	12
b d d	13
c d a	14
c d b	15
c d c	16

What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language V (for *Vibrating*)

$\mathcal{V}$

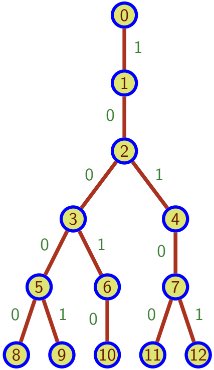


$\langle 10 \rangle_V = b\,d\,a$

.	0
b	1
c	2
d	3
b d	4
c d	5
d a	6
d b	7
d c	8
d d	9
b d a	10
b d b	11
b d c	12
b d d	13
c d a	14
c d b	15
c d c	16

What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model
- A general working hypothesis:
  Prefix-Closed Extendable Languages



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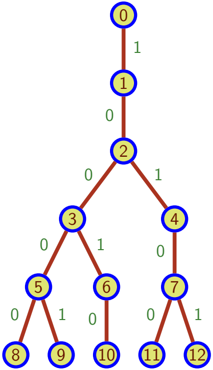
$L \subseteq A^*$ an ANS



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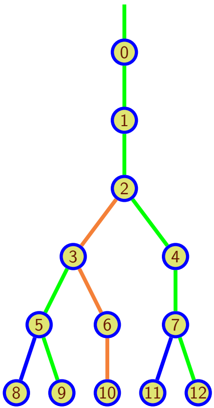

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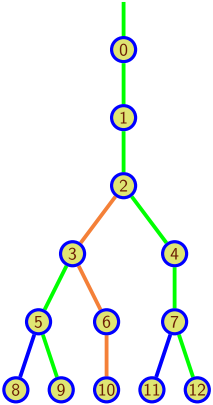
Notation

$$\mathbf{u}_L(\ell) = \text{card}\left(L \cap A^\ell\right)$$
$$\mathbf{v}_L(\ell) = \text{card}\left(L \cap A^{\leqslant \ell}\right) = \sum_{i=0}^{\ell} \mathbf{u}_L(i)$$



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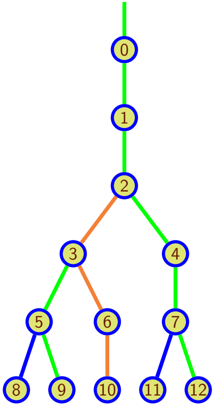

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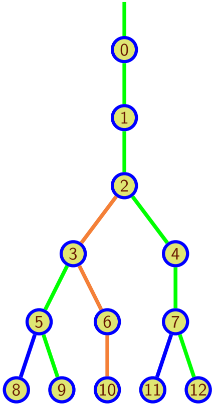

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requires L *prefix-closed* and *extendable*

i.e. to be a PCE language

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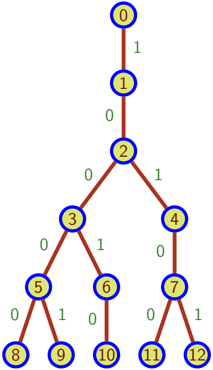

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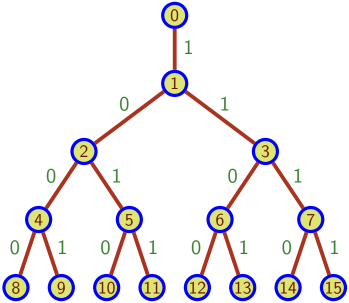
Fact: *'All' 'classical' ANS are* PCE

What we learn from the primal observation: an hypothesis

'All' 'classical' ANS are PCE

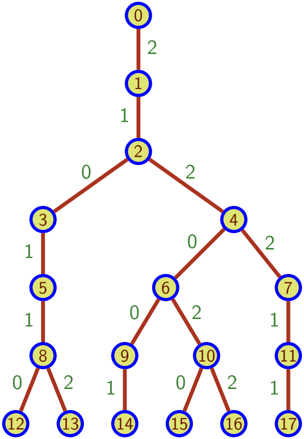


L_2

.
1
1 0
1 1
1 0 0
1 0 1
1 1 0
1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
1 0 0 0 0

What we learn from the primal observation: an hypothesis

'All' 'classical' ANS are PCE

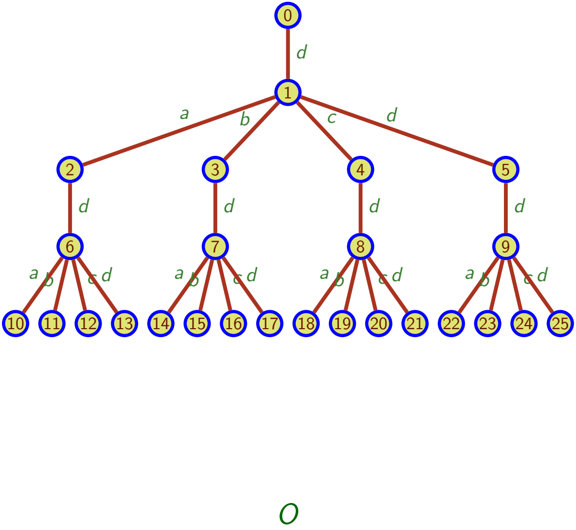


$L_{\frac{3}{2}}$

.
2
2 1
2 1 0
2 1 2
2 1 0 1
2 1 2 0
2 1 2 2
2 1 0 1 1
2 1 2 0 0
2 1 2 0 2
2 1 2 2 1
2 1 0 1 1 0
2 1 0 1 1 2
2 1 2 0 0 1
2 1 2 0 2 0
2 1 2 0 2 2
2 1 2 2 1 1

What we learn from the primal observation: an hypothesis

The ANS we consider are PCE



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d c
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d b d
d c d
d d d
d a d a
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d a d c
d a d d
d b d a
d b d b
d b d c

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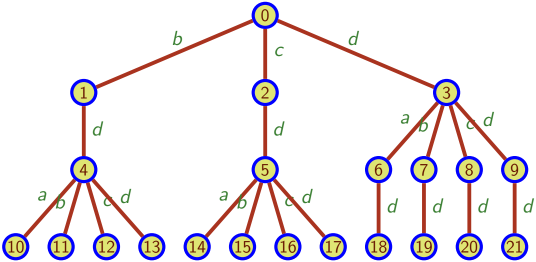

V

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b
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d b
d c
d d
b d a
b d b
b d c
b d d
c d a
c d b
c d c

What we learn from the primal observation: a new parameter

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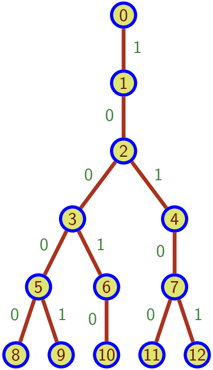
- An essential parameter: the local growth rate



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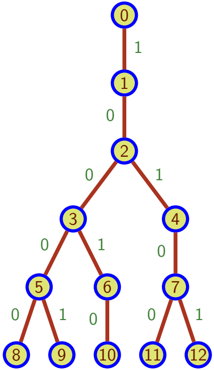

From $$\sum_{i=\mathbf{v}_L(\ell-1)}^{\mathbf{v}_L(\ell)-1} \mathrm{cp}_L(i) = \mathbf{v}_L(\ell)$$

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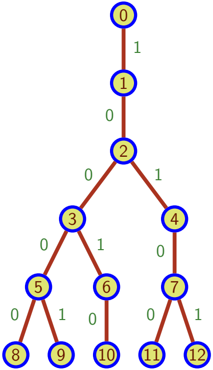
follows $$\sum_{i=0}^{\mathbf{v}_L(\ell)-1} \mathrm{cp}_L(i) = \sum_{j=0}^{\ell} \mathbf{v}_L(j)$$



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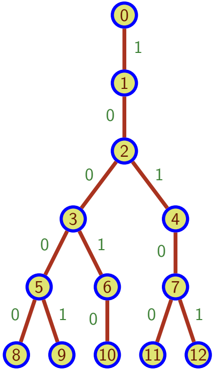

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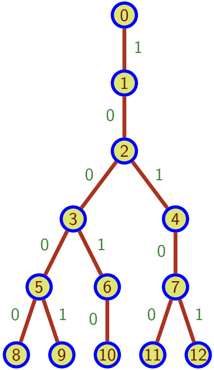

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then $$\mathrm{CP}_L = \lim_{\ell\to\infty} \frac{1}{\mathbf{v}_L(\ell)} \sum_{j=0}^{\ell} \mathbf{v}_L(j)$$ exists

Intermede: a freshperson calculus lemma

Lemma

$$(x_\ell)_{\ell\in\mathbb{N}} \quad x_\ell \in \mathbb{R}_+ \qquad \forall \ell \quad y_\ell = \sum_{j=0}^{\ell-1} x_j \qquad \gamma > 1$$

TFAE

(i) $\lim_{\ell\to\infty} \frac{x_{\ell+1}}{x_\ell} = \gamma$

(ii) $\lim_{\ell\to\infty} \frac{y_{\ell+1}}{y_\ell} = \gamma$

(iii) $\lim_{\ell\to\infty} \frac{y_\ell}{x_\ell} = \frac{\gamma}{\gamma - 1}$

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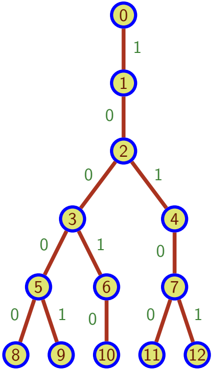

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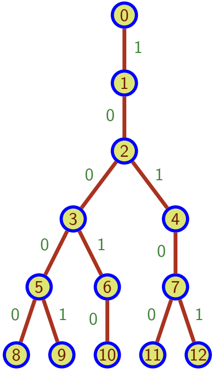

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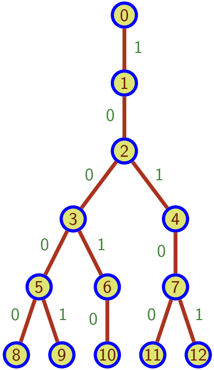

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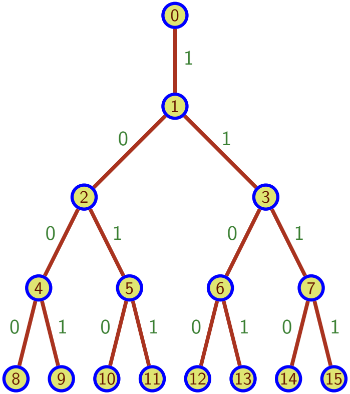

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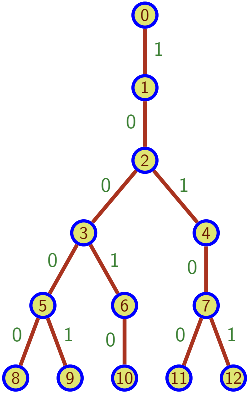
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
1 0 0 0 0

L_2 $\qquad \lim_{\ell \to \infty} \frac{\mathbf{u}_2(\ell+1)}{\mathbf{u}_2(\ell)} = \gamma_2 = 2$

What we learn from the primal observation: a new parameter



.
1
1 0
1 0 0
1 0 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 0 0 0
1 0 0 0 1
1 0 0 1 0
1 0 1 0 0
1 0 1 0 1
1 0 0 0 0 0
1 0 0 0 0 1
1 0 0 0 1 0
1 0 0 1 0 0

L_F $\qquad$ $\lim_{\ell\to\infty} \frac{\mathbf{u}_F(\ell+1)}{\mathbf{u}_F(\ell)} = \gamma_F = \varphi$

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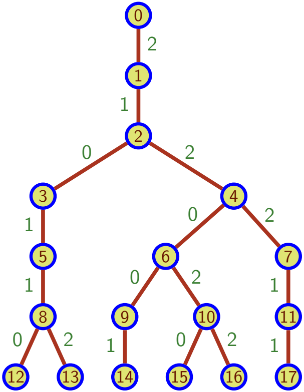

.
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2 1
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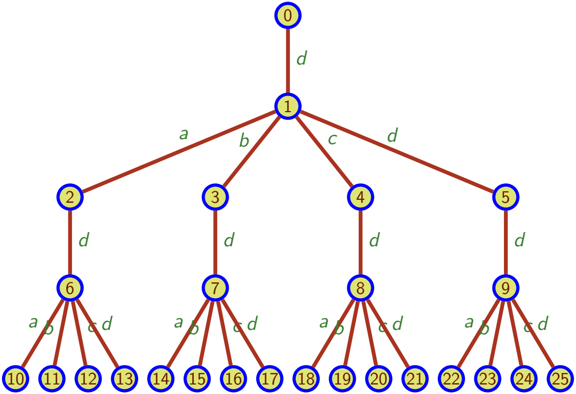
2 1 2
2 1 0 1
2 1 2 0
2 1 2 2
2 1 0 1 1
2 1 2 0 0
2 1 2 0 2
2 1 2 2 1
2 1 0 1 1 0
2 1 0 1 1 2
2 1 2 0 0 1
2 1 2 0 2 0
2 1 2 0 2 2

$L_{\frac{3}{2}}$

$$\lim_{\ell\to\infty} \frac{\mathbf{u}_{\frac{3}{2}}(\ell+1)}{\mathbf{u}_{\frac{3}{2}}(\ell)} = \gamma_{\frac{3}{2}} = \frac{3}{2}$$

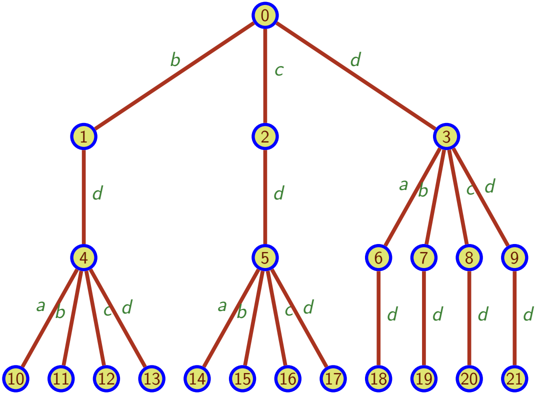
What we learn from the primal observation: a new parameter



.
d
d a
d b
d c
d d
d a d
d b d
d c d
d d d
d a d a
d a d b
d a d c
d a d d
d b d a
d b d b
d b d c

O $\qquad \dfrac{\mathbf{u}_O(2\ell+1)}{\mathbf{u}_O(\ell)} = 1 \qquad \dfrac{\mathbf{u}_O(2\ell+2)}{\mathbf{u}_O(2\ell+1)} = 4$

What we learn from the primal observation: a new parameter



V $\mathbf{u}_V(\ell) = 3\,2^{\ell-1}$ $\gamma_V = 2$

.
b
c
d
b d
c d
d a
d b
d c
d d
b d a
b d b
b d c
b d d
c d a
c d b
c d c

A natural question

Proposition

If $\quad \mathrm{CP}_L = \lim_{N\to\infty} \dfrac{1}{N} \displaystyle\sum_{i=0}^{N-1} \mathrm{cp}_L(i) \quad$ *exists,*

then the local growth rate $\quad \lim_{\ell\to\infty} \dfrac{\mathbf{u}_L(\ell+1)}{\mathbf{u}_L(\ell)} = \gamma_L \quad$ *exists*

and $\quad \mathrm{CP}_L = \dfrac{\gamma_L}{\gamma_L - 1}$

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Proposition

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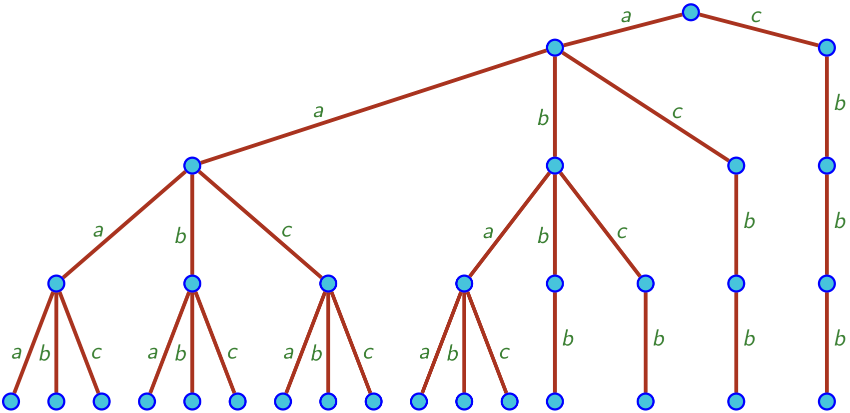
and $\mathsf{CP}_L = \frac{\gamma_L}{\gamma_L - 1}$

Question

Is the existence of the local growth rate sufficient to insure the existence of the carry propagation?

An unbalanced tree

$U \subseteq \{a, b, c\}^*$

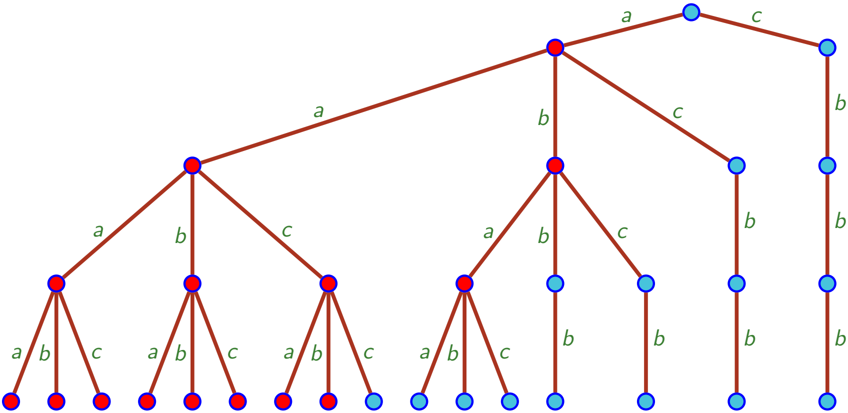


$\mathbf{u}_U(\ell) = 2^\ell$

$$\frac{\mathbf{u}_U(\ell+1)}{\mathbf{u}_U(\ell)} = \gamma_U = 2$$

An unbalanced tree

$U \subseteq \{a, b, c\}^*$



$$\mathbf{u}_U(\ell) = 2^\ell \qquad \frac{\mathbf{u}_U(\ell+1)}{\mathbf{u}_U(\ell)} = \gamma_U = 2$$

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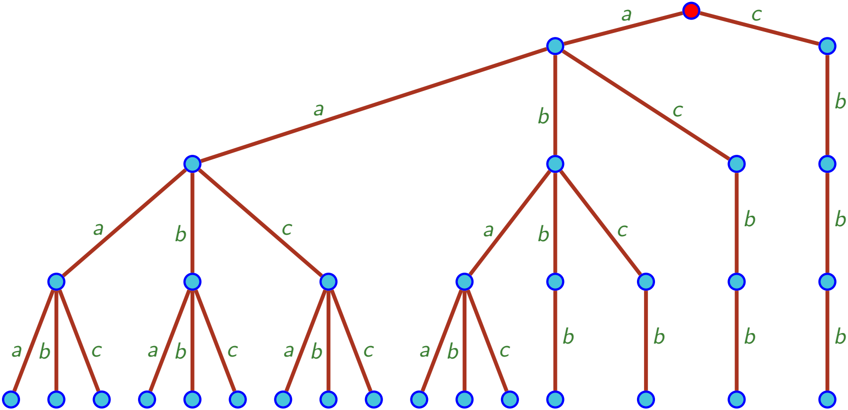

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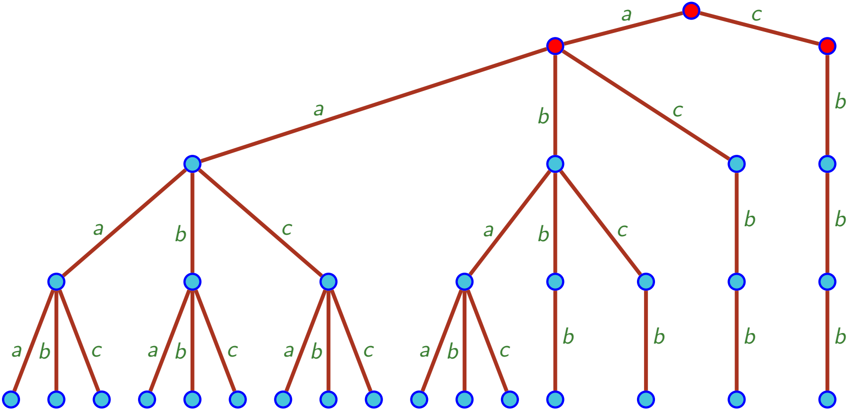

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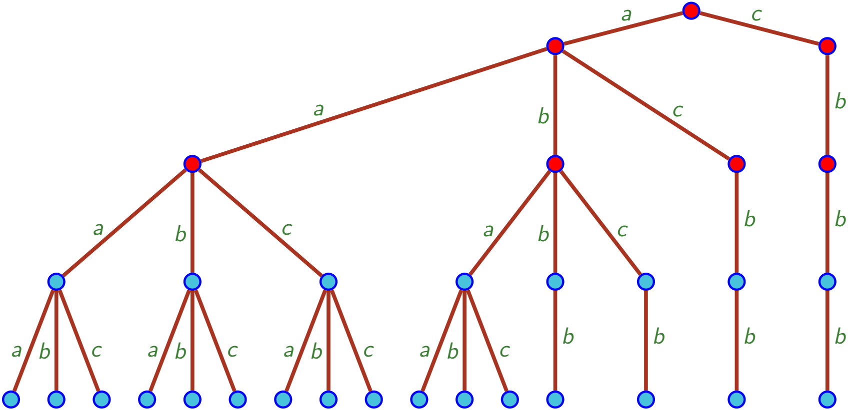

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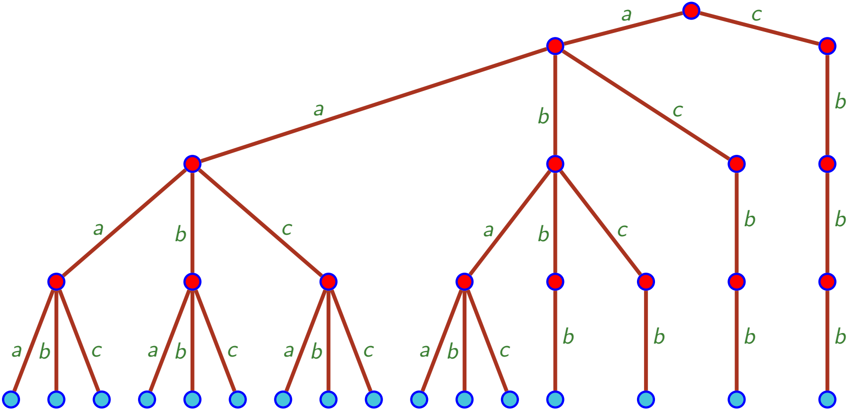
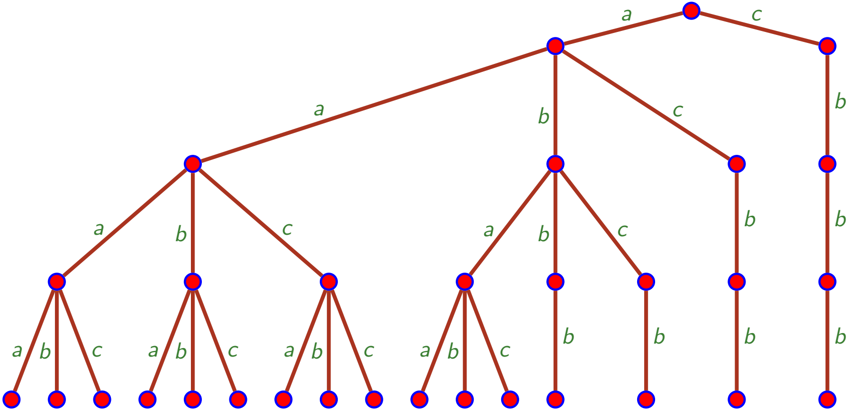


An unbalanced tree

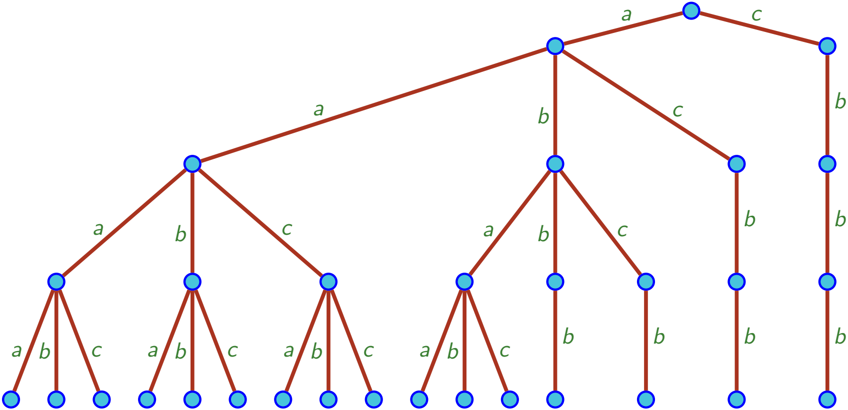
$U \subseteq \{a, b, c\}^*$



$$\lim_{\ell \to \infty} \frac{1}{\mathbf{v}_U(\ell)} \sum_{j=0}^{\mathbf{v}_U(\ell)-1} \mathrm{cp}_U(j) = 2$$

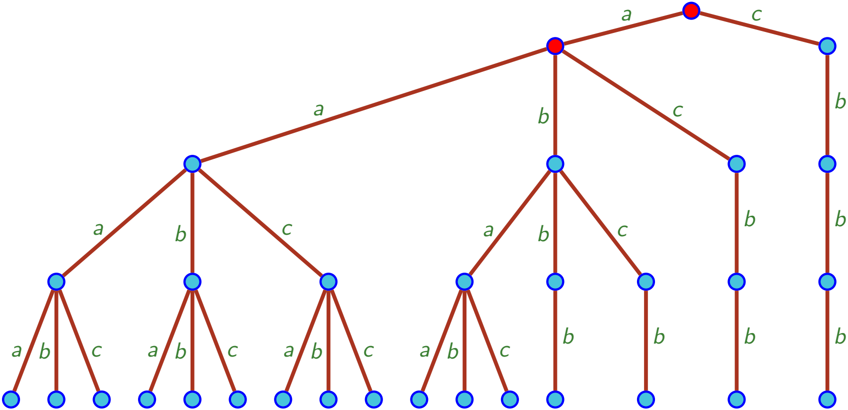
An unbalanced tree

$U \subseteq \{a, b, c\}^*$



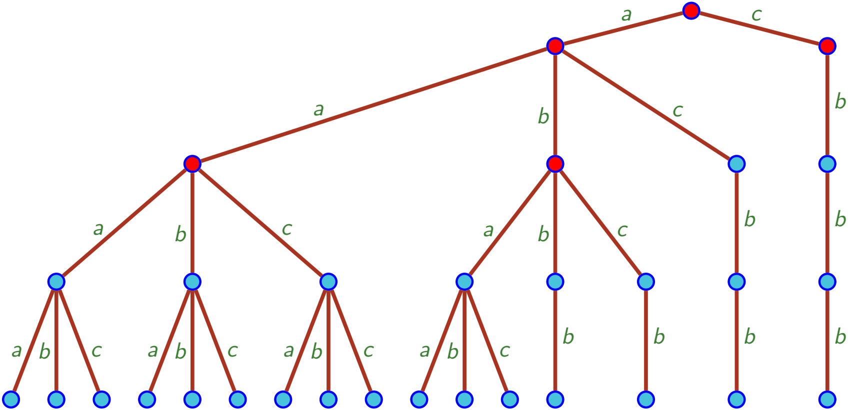
An unbalanced tree

$U \subseteq \{a, b, c\}^*$



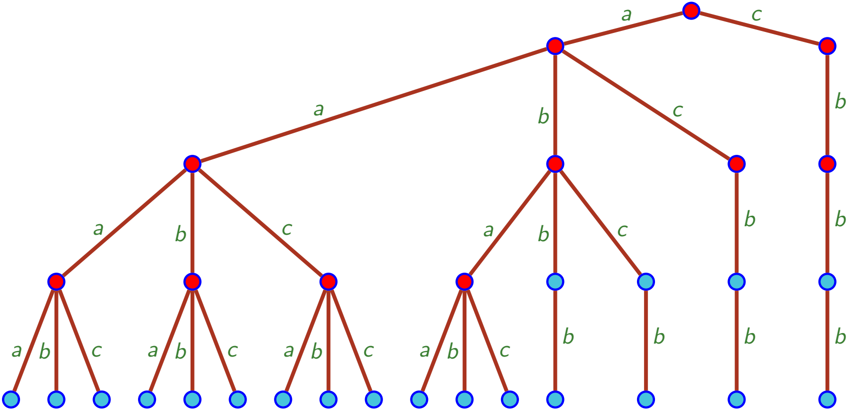
An unbalanced tree

$U \subseteq \{a, b, c\}^*$



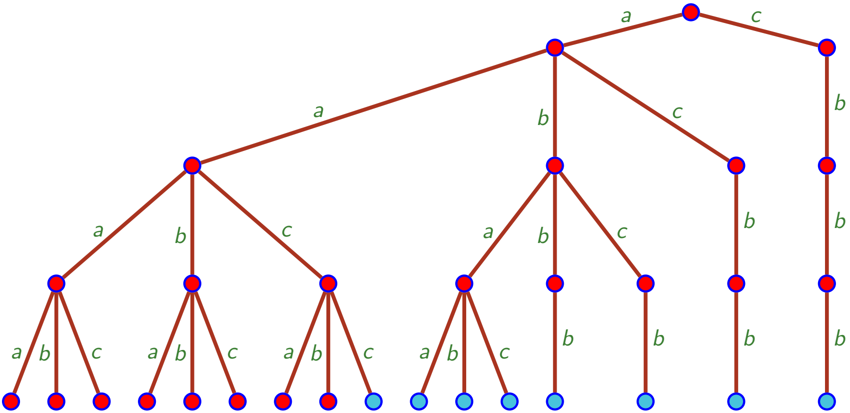
An unbalanced tree

$U \subseteq \{a, b, c\}^*$



An unbalanced tree

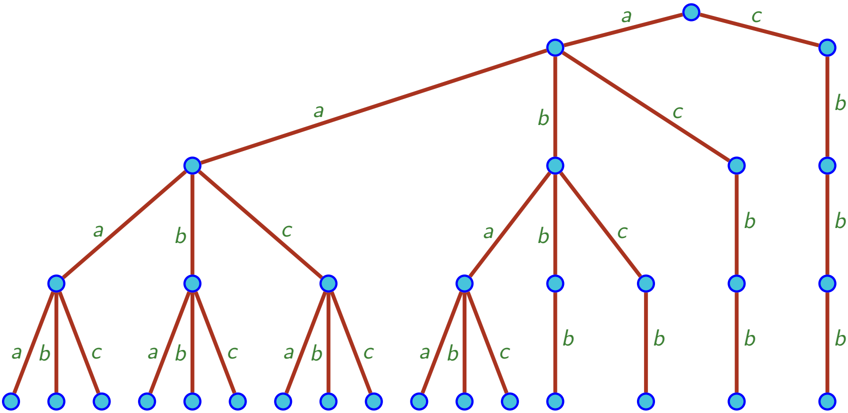
$U \subseteq \{a, b, c\}^*$



$$\lim_{\ell\to\infty} \frac{1}{\mathbf{v}'_U(\ell)} \sum_{j=0}^{\mathbf{v}'_U(\ell)-1} \mathrm{cp}_U(j) = \frac{11}{6} \neq 2$$

An unbalanced tree

$U \subseteq \{a, b, c\}^*$

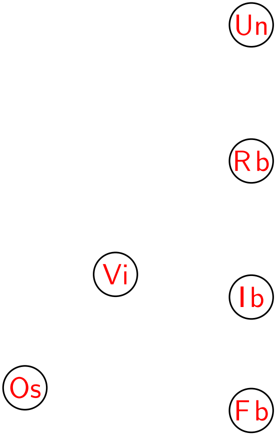


$$\mathrm{CP}_U = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_U(i)$$ does not exist

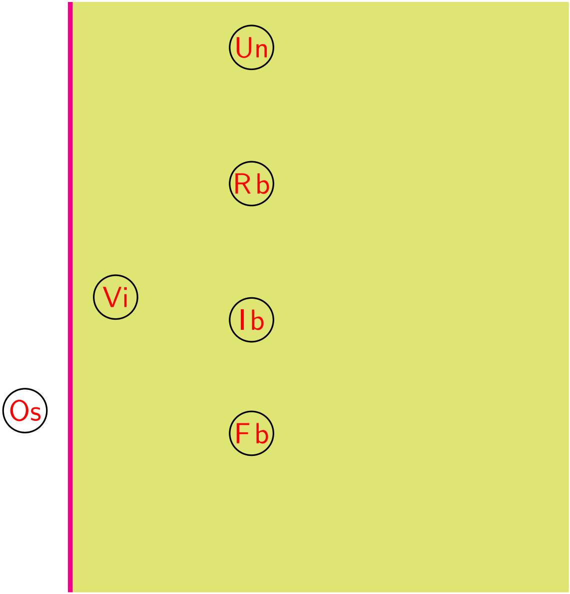
A first conclusion

The *existence* of the carry propagation
is more difficult to prove
than the *computation* of the carry propagation itself

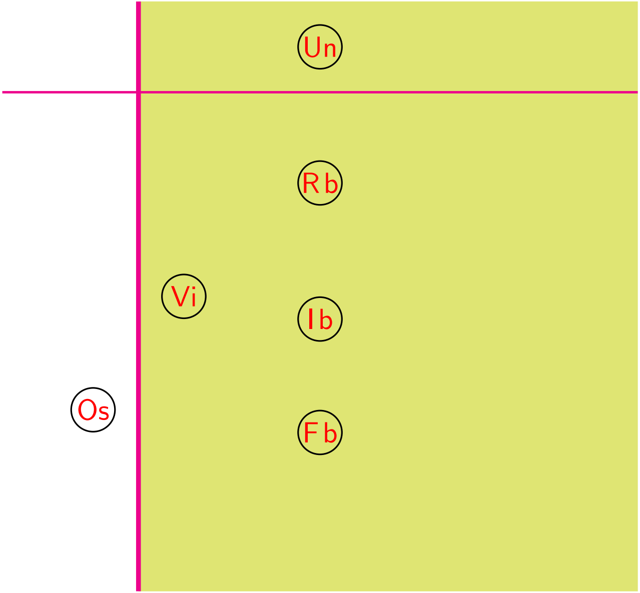
Roadmap



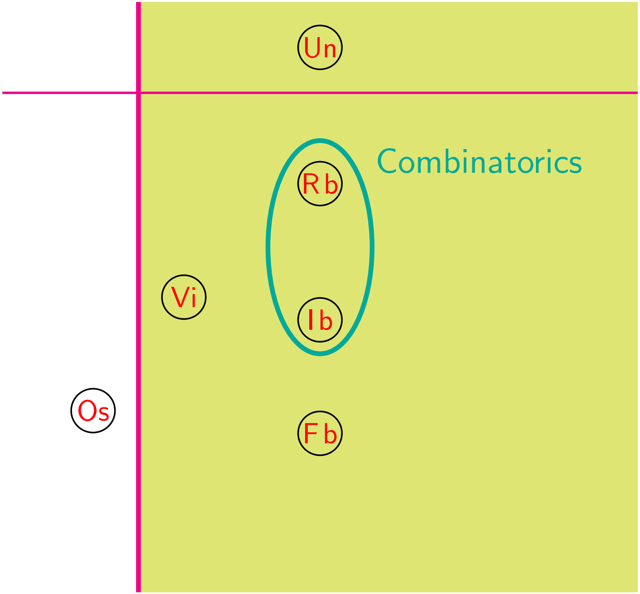
Roadmap



Roadmap



Roadmap



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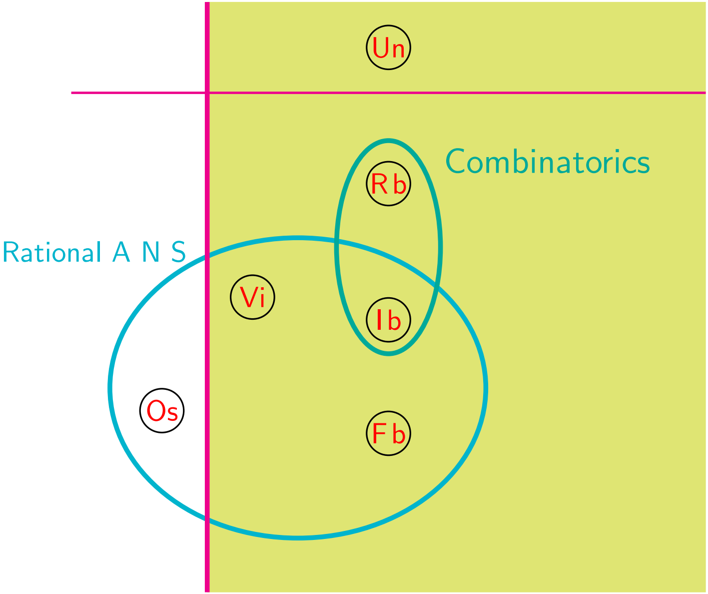
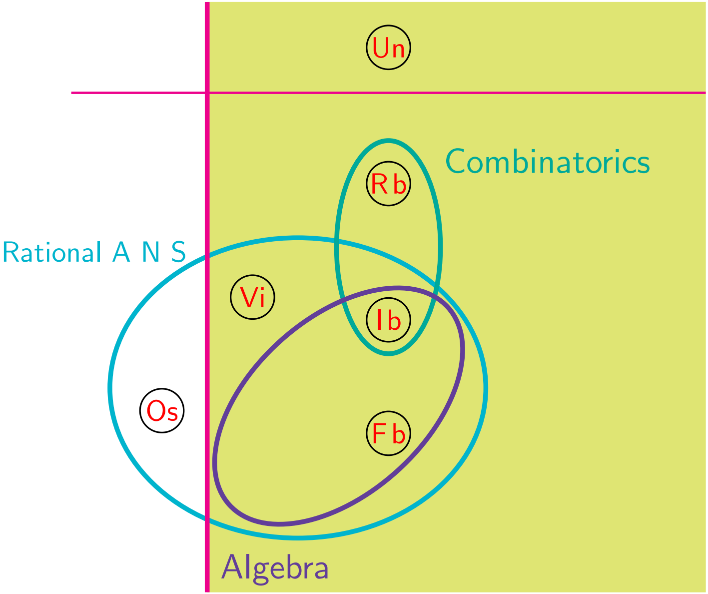
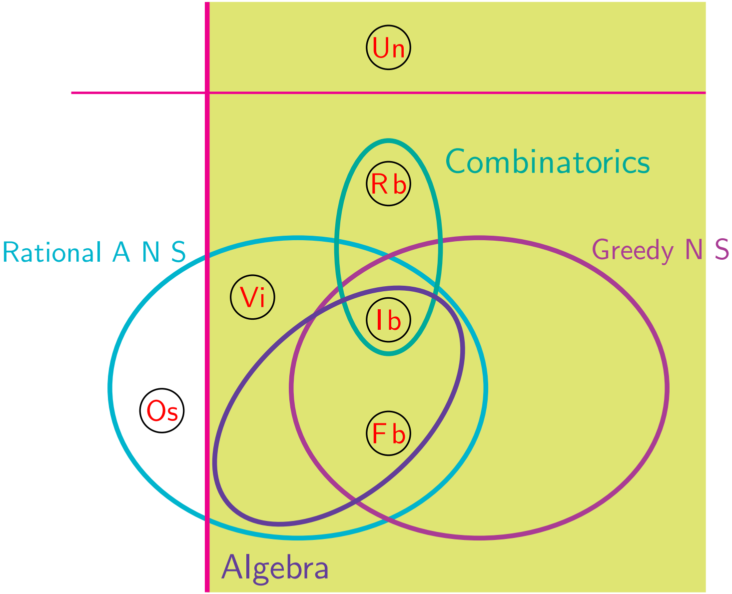


Roadmap

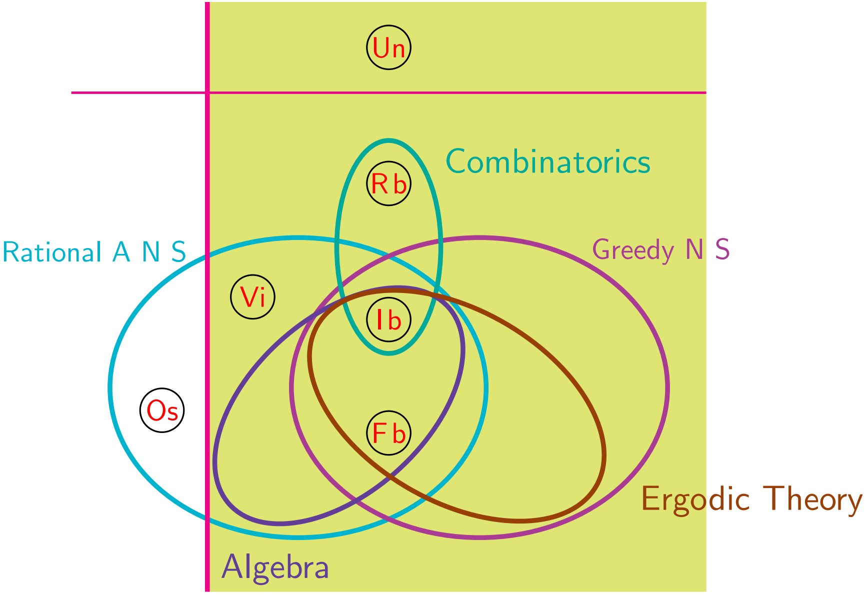


Roadmap

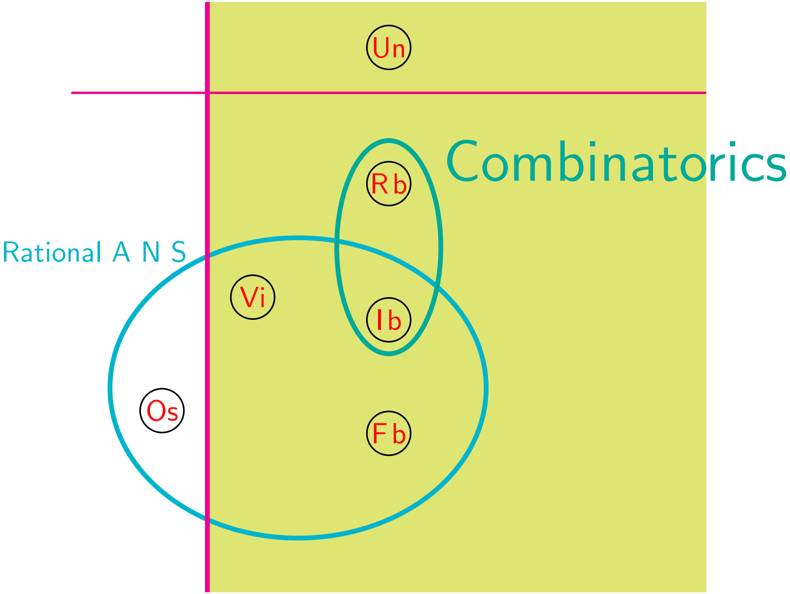


Un
Combinatorics
Rb
Rational A N S
Greedy N S
Vi
Ib
Os
Fb
Algebra

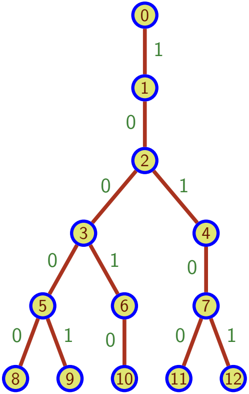
Roadmap



Un
Combinatorics
Rb
Rational A N S
Greedy N S
Vi
Ib
Os
Fb
Ergodic Theory
Algebra

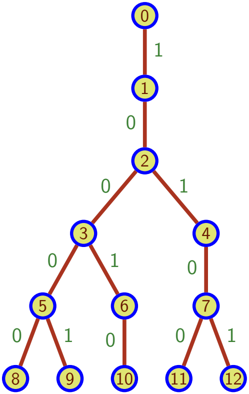

Un
Combinatorics
Rb
Rational A N S
Vi
Ib
Os
Fb

A definition: the signature of a language

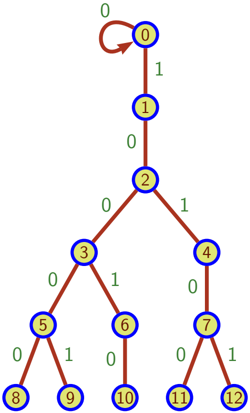


A tree (base Fibonacci)

A definition: the signature of a language

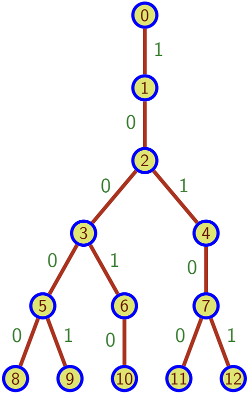


A tree (base Fibonacci)



An i-tree

A definition: the signature of a language



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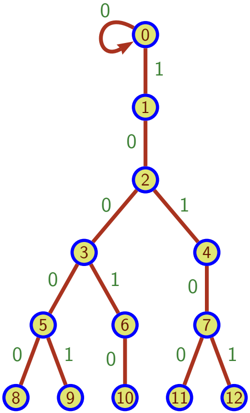
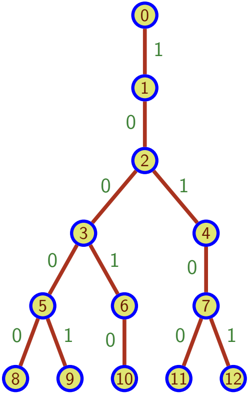


An i-tree

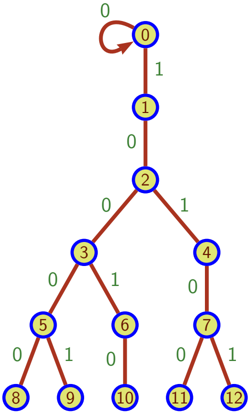
Definition

Signature of L = sequence of the *degrees of the nodes* of the *i-tree* of L in a *breadth first traversal*.

A definition: the signature of a language



A tree (base Fibonacci)



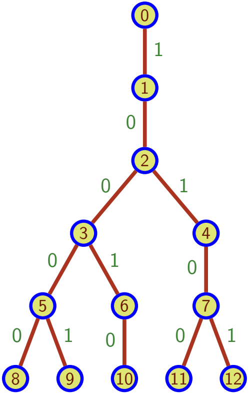
An i-tree

Definition

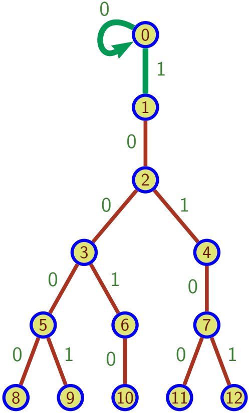
Signature of L = sequence of the *degrees of the nodes* of the *i-tree* of L in a *breadth first traversal*.

$\boldsymbol{s}_F =$

A definition: the signature of a language



A tree (base Fibonacci)



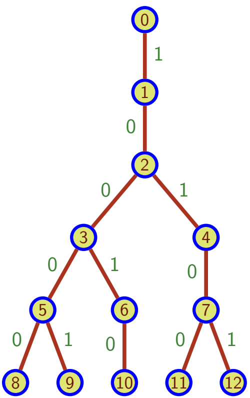
An i-tree

Definition

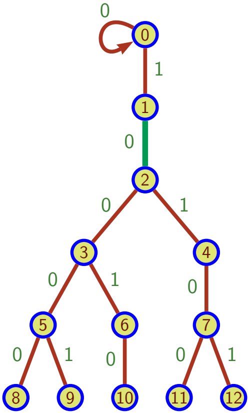
Signature of L = sequence of the *degrees of the nodes* of the *i-tree* of L in a *breadth first traversal*.

$$s_F = 2$$

A definition: the signature of a language



A tree (base Fibonacci)



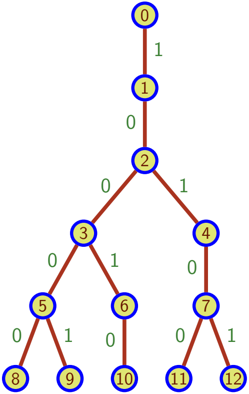
An i-tree

Definition

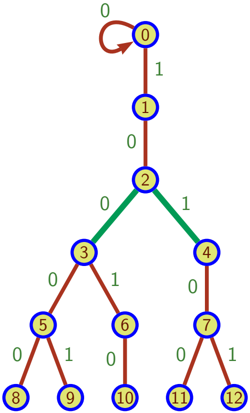
Signature of L = sequence of the *degrees of the nodes* of the *i-tree* of L in a *breadth first traversal*.

$$s_F = 2 \; 1$$

A definition: the signature of a language



A tree (base Fibonacci)



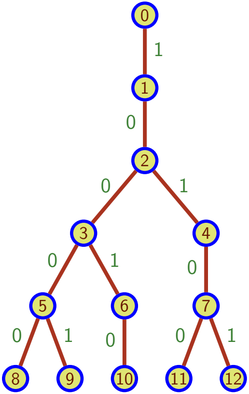
An i-tree

Definition

Signature of L = sequence of the *degrees of the nodes* of the *i-tree* of L in a *breadth first traversal*.

$\boldsymbol{s}_F = 2\ 1\ 2$

A definition: the signature of a language



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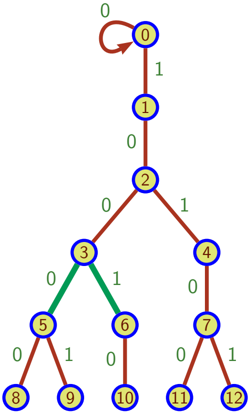

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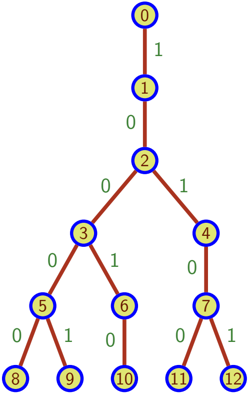
$\boldsymbol{s}_F = 2 \;\; 1 \;\; 2 \;\; 2$

A definition: the signature of a language



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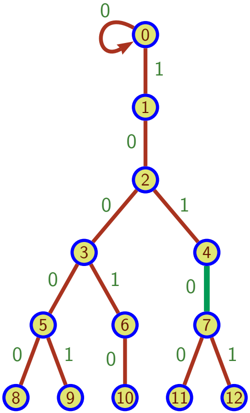

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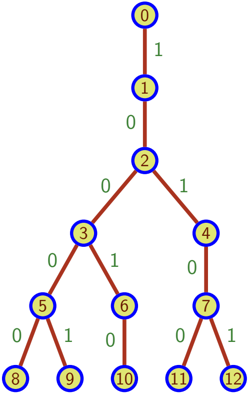
$$\boldsymbol{s}_F = 2\ 1\ 2\ 2\ 1$$

A definition: the signature of a language



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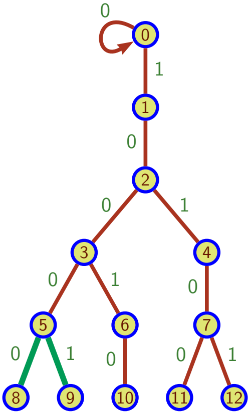

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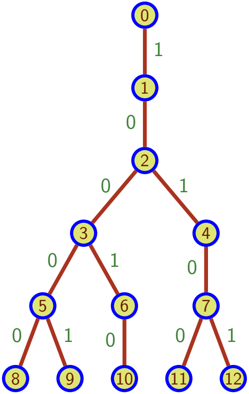

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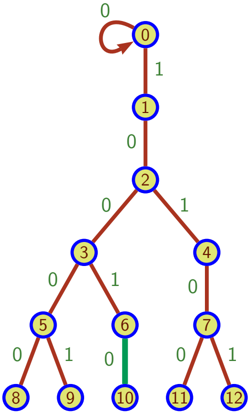

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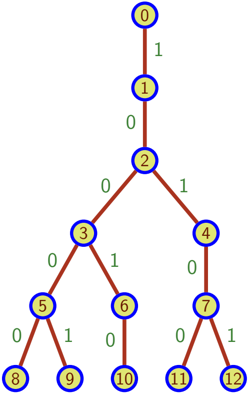
A definition: the signature of a language



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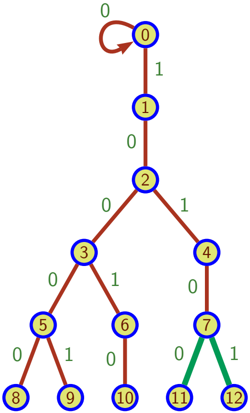


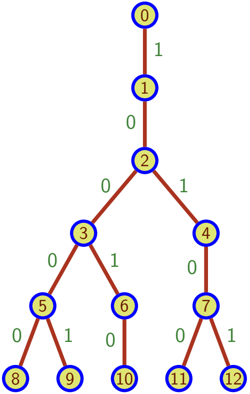
An i-tree

Definition

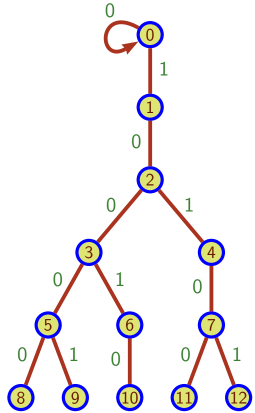
Signature of L = sequence of the *degrees of the nodes* of the *i-tree* of L in a *breadth first traversal*.

$$s_F = 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2$$

A definition: the signature of a language



A tree (base Fibonacci)



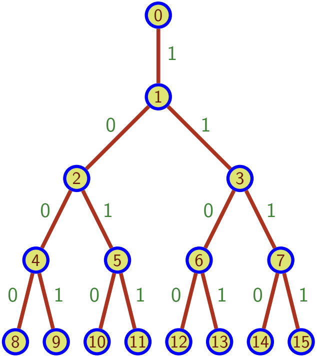
An i-tree

Definition

Signature of L $=$ sequence of the *degrees of the nodes* of the *i-tree* of L in a *breadth first traversal*.

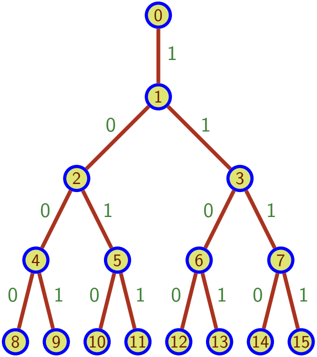
$$s_F = 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ \cdots$$

A definition: the signature of a language



Another tree (base 2)

A definition: the signature of a language



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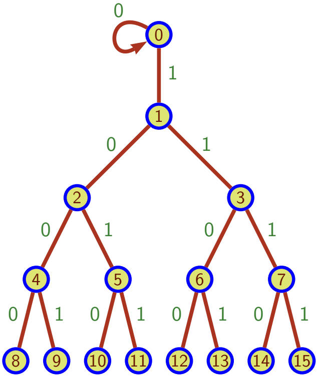
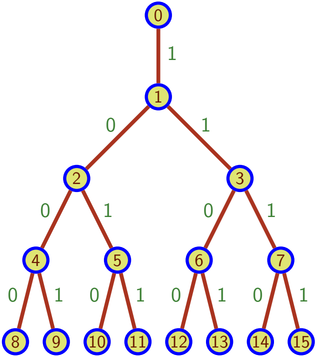
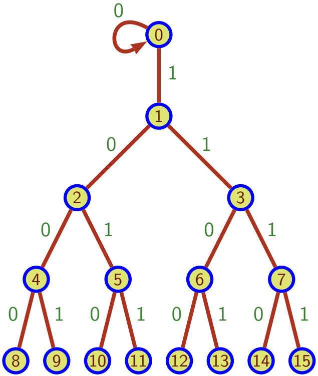


Another i-tree

A definition: the signature of a language



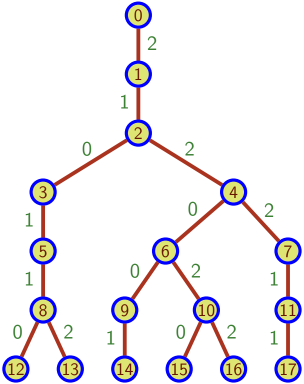
Another tree (base 2)



Another i-tree

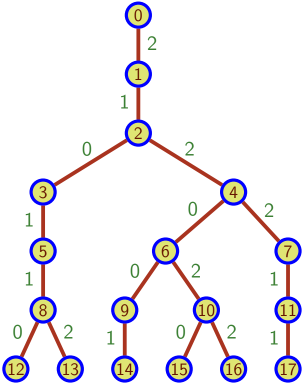
$$\boldsymbol{s}_2 = \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; 2 \; \cdots$$

A definition: the signature of a language

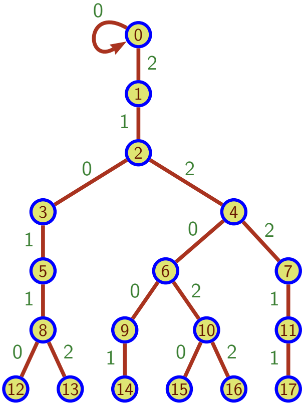


Another tree (base $\frac{3}{2}$)

A definition: the signature of a language

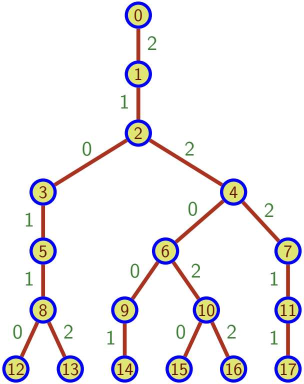


Another tree (base $\frac{3}{2}$)

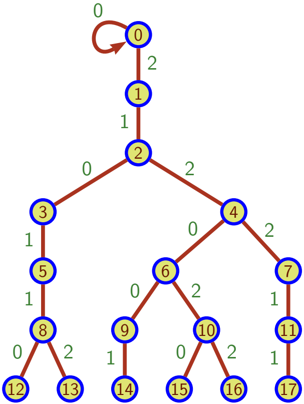


Another i-tree

A definition: the signature of a language



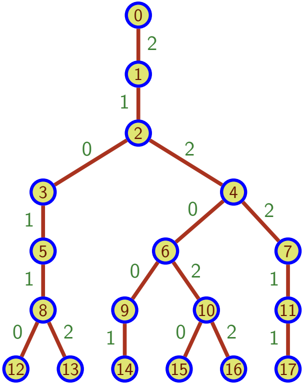
Another tree(base $\frac{3}{2}$)



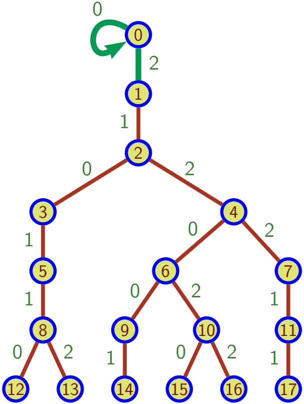
Another i-tree

$\mathbf{s}_{\frac{3}{2}} =$

A definition: the signature of a language



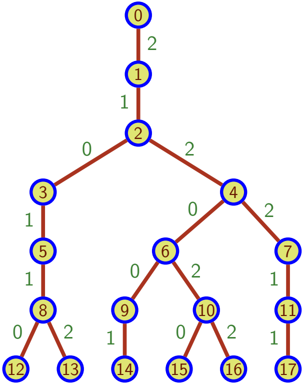
Another tree(base $\frac{3}{2}$)



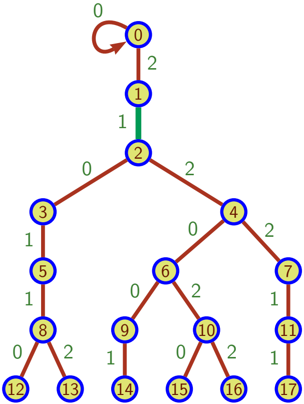
Another i-tree

$$s_{\frac{3}{2}} = 2$$

A definition: the signature of a language



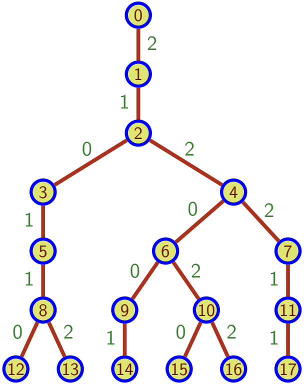
Another tree(base $\frac{3}{2}$)



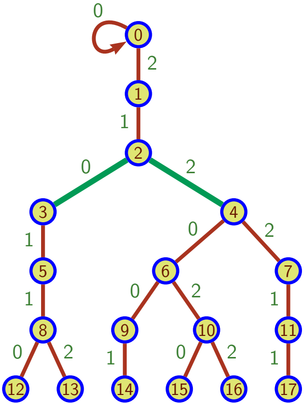
Another i-tree

$\boldsymbol{s}_{\frac{3}{2}} = 2 \ 1$

A definition: the signature of a language



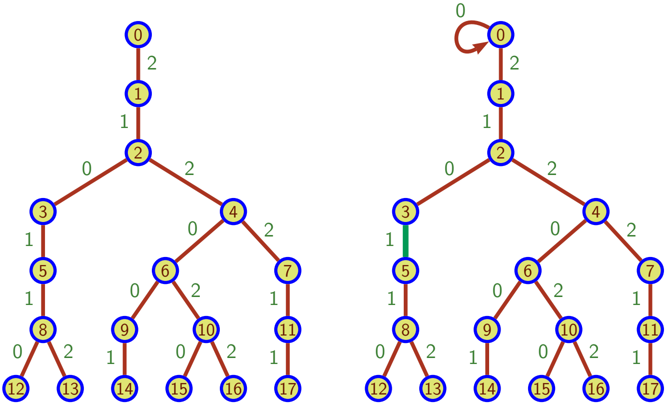
Another tree(base $\frac{3}{2}$)



Another i-tree

$s_{\frac{3}{2}} = 2\ 1\ 2$

A definition: the signature of a language

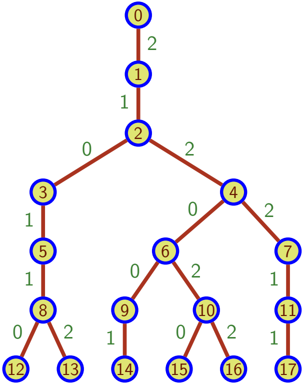


Another tree(base $\frac{3}{2}$)

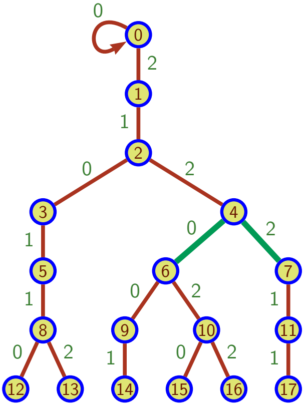
Another i-tree

$$\boldsymbol{s}_{\frac{3}{2}} = 2\ \ 1\ \ 2\ \ 1$$

A definition: the signature of a language



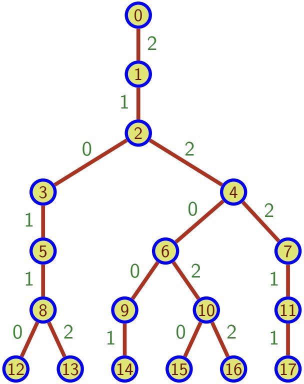
Another tree(base $\frac{3}{2}$)



Another i-tree

$$s_{\frac{3}{2}} = 2 \; 1 \; 2 \; 1 \; 2$$

A definition: the signature of a language



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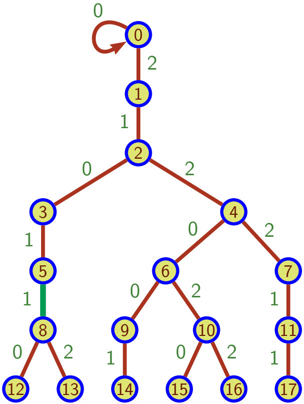

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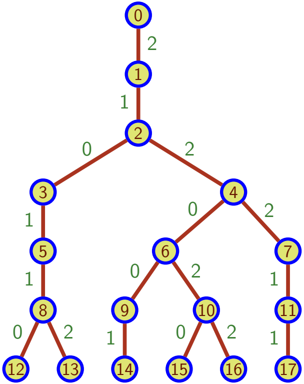

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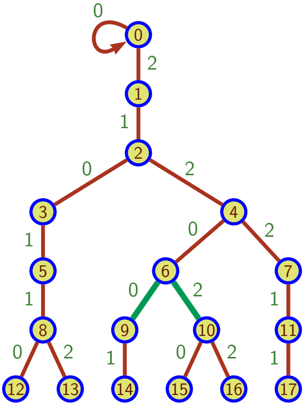

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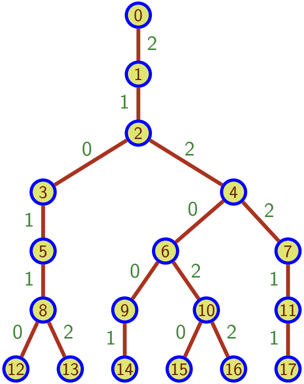

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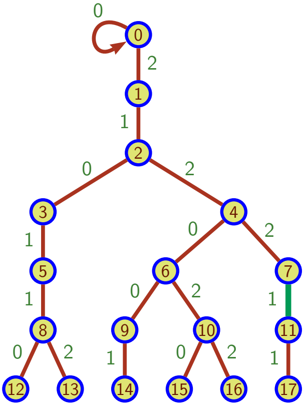

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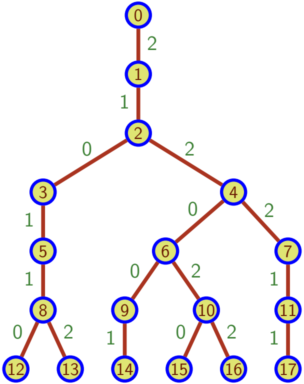

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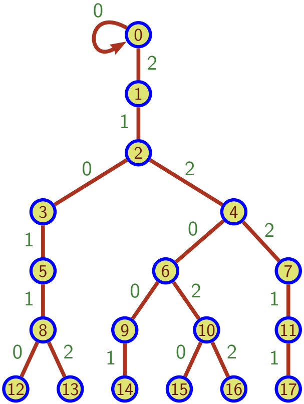

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A definition: the signature of a language

Definition

A sequence $\boldsymbol{s} = s_0\, s_1\, s_1 \cdots$ is *valid* if:

$$\forall j \in \mathbb{N} \qquad \sum_{i=0}^{j} s_i > j + 1 \;\; .$$

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Proposition

The signature of an infinite PCE *language is valid and a valid signature uniquely defines an (i)-tree.*

Periodic signature

p, q coprime integers $p > q \geqslant 1$

Definition

1. $\mathbf{r}$ rhythm of directing parameter (q, p)

$$\mathbf{r} = (r_0, r_1, \ldots, r_{q-1}) \qquad \sum_{i=0}^{q-1} r_i = p$$

2. A purely periodic signature

$$s = \mathbf{r}^{\omega}$$

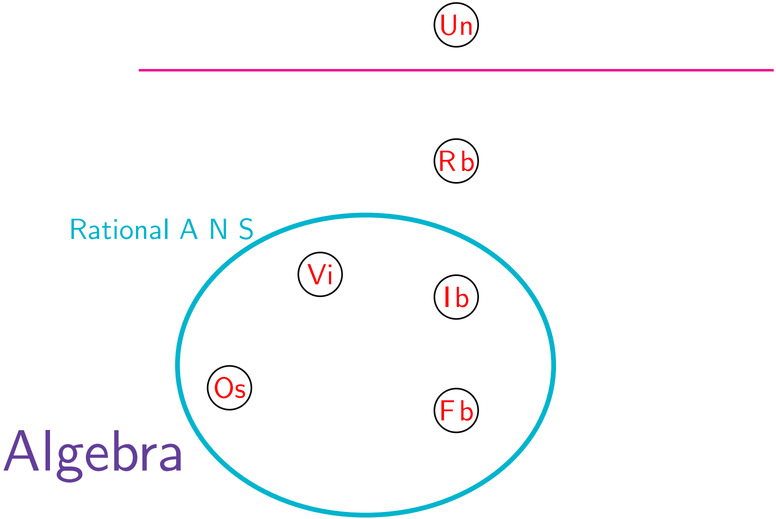
Proposition (Marsault-S. 17)

The signature of $L_{\frac{p}{q}}$ *is periodic*
and its period is a rhythm of parameter (q, p) *.*

Theorem

L PCE *with ultimately periodic signature*
with rhythm of parameter (q, p) *.*

Then CP_L *exists and* $\mathrm{CP}_L = \dfrac{p}{p-q}$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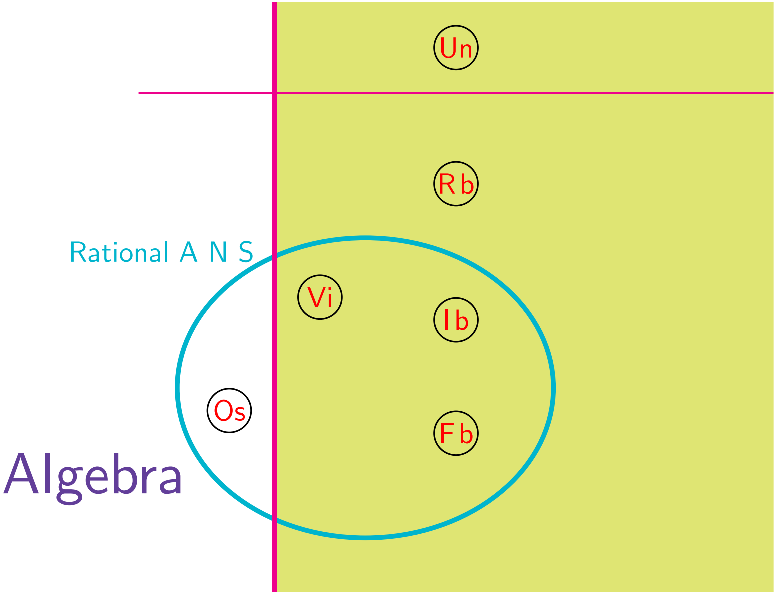
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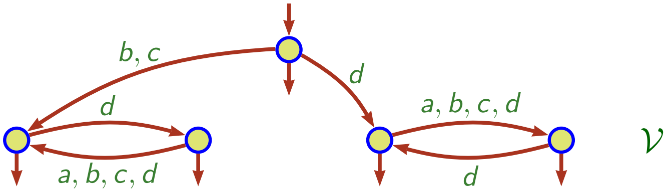
Surprise !



$$\mathbf{u}_V(\ell) = 3\,.\,2^{\ell-1}$$

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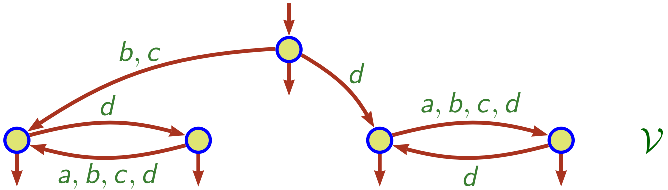

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$$\liminf_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \mathrm{cp}_V(i) \;\leqslant\; \frac{28}{15} \;<\; \frac{13}{6} \;\leqslant\; \limsup_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \mathrm{cp}_V(i)$$

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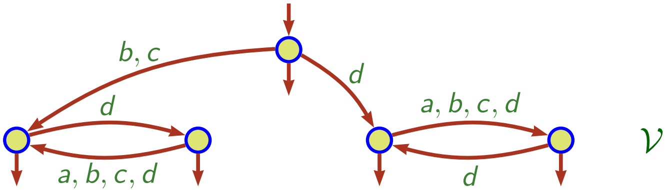

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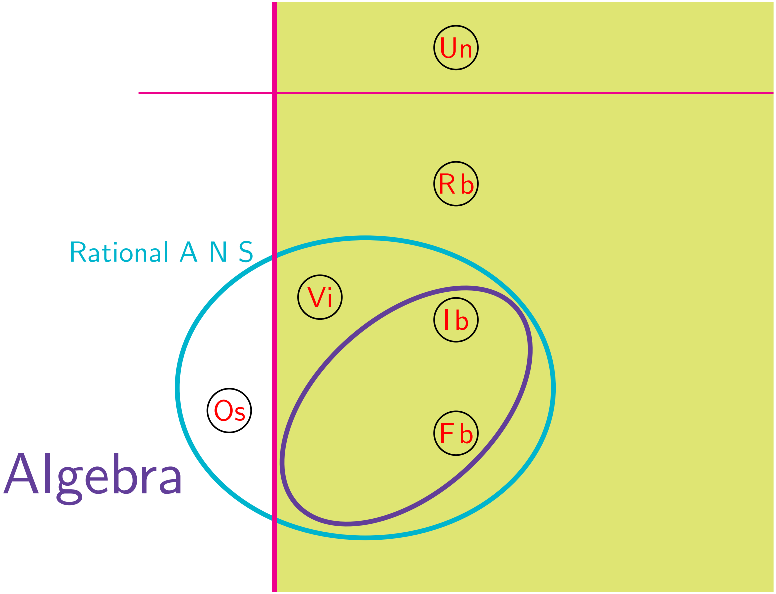
γ_V exists but CP_V does not exists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Generating functions

Definition

$L \subseteq A^*$ $\quad \mathrm{g}_L(z)$ *generating function* of L

$$\mathrm{g}_L(z) = \sum_{\ell=0}^{\infty} \mathbf{u}_L(\ell)\, z^{\ell}$$

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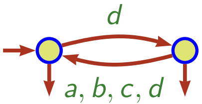
Cayley-Hamilton Theorem $\Longrightarrow$ $\mathbf{u}_L(\ell)$ satisfy a *linear recurrence relation*

defined by $\mathrm{P}_{\mathcal{A}}$, *characteristic polynomial* of $M_{\mathcal{A}}$

Some examples

$$M_{\mathcal{O}} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \qquad \mathsf{P}_{\mathcal{O}} = X^2 - 4$$

$$\mathbf{u}_{\mathcal{O}}(\ell) = \frac{3}{4}\,2^{\ell} + \frac{1}{4}\,(-2)^{\ell}$$



$\mathcal{O}$

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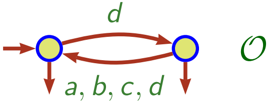

$$M_{\mathcal{C}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad \mathrm{P}_{\mathcal{C}} = X^2 - 4$$

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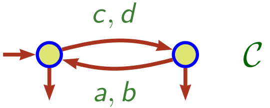

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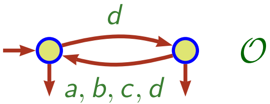

$$M_{\mathcal{C}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad \mathrm{P}_{\mathcal{C}} = X^2 - 4$$

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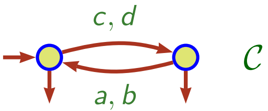
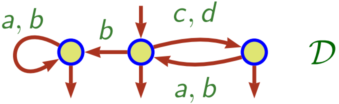


$$M_{\mathcal{D}} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathrm{P}_{\mathcal{D}} = (X^2-4)(2-X)$$

$$\mathbf{u}_{D}(\ell) = \left(\frac{1}{4}\,\ell + \frac{7}{8}\right)2^{\ell} + \frac{1}{8}\,(-2)^{\ell}$$



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The *eigenvalues* of L are the zeroes λ_1, λ_2,..., λ_t of P_L and

$$\forall \ell \in \mathbb{N} \qquad \mathbf{u}_L(\ell) = \sum_{j=1}^{t} \lambda_j^{\ell}\, P_j(\ell)$$

where $\deg P_j$ $=$ multiplicity of λ_j in P_L minus 1

Positive rational functions

Theorem (Berstel 71)

$f(z)$ $\mathbb{R}_+$ *-rational function (not a polynomial)*

λ *maximum of the moduli of its eigenvalues.*

(i) λ *is an eigenvalue of* $f(z)$ *(hence an eigenvalue in* $\mathbb{R}_+$ *)*

(ii) *Every eigenvalue of* $f(z)$ *of modulus* λ *is of the form* $\lambda \mathrm{e}^{i\,\theta}$ *, where* $\mathrm{e}^{i\,\theta}$ *is a root of the unity*

(iii) *The multiplicity of any eigenvalue of modulus* λ *is at most that of* λ

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(iii) *The multiplicity of any eigenvalue of modulus λ is at most that of λ*

Definition

(i) $f(z)$ is DEV if λ is the *only* eigenvalue of modulus λ

(ii) $f(z)$ is ADEV if the multiplicity of λ is *greater* than the multiplicity of the other eigenvalues of modulus λ

Examples

- O is neither DEV nor ADEV $\quad \mathbf{u}_O(\ell) = \frac{3}{4}\,2^\ell + \frac{1}{4}\,(-2)^\ell$
- V is DEV $\quad \mathbf{u}_V(\ell) = \frac{3}{2}\,2^\ell$
- D is ADEV but not DEV $\quad \mathbf{u}_D(\ell) = (\frac{1}{4}\,\ell + \frac{7}{8})\,2^\ell + \frac{1}{8}\,(-2)^\ell$

Theorem

A rational language L is ADEV *iff the local growth rate γ_L exists.*

In this case, the modulus of L is equal to γ_L .

Theorem

L ADEV *rational* PCE *and* λ *its modulus.*

If every quotient of L *whose modulus is equal to* λ *is* ADEV,

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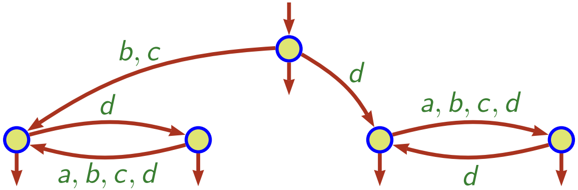
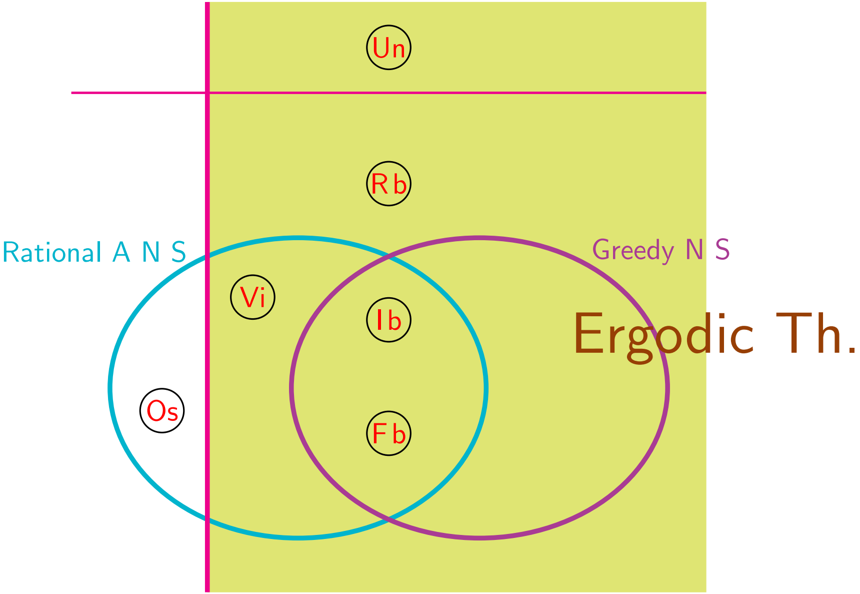
$\mathcal{V}$





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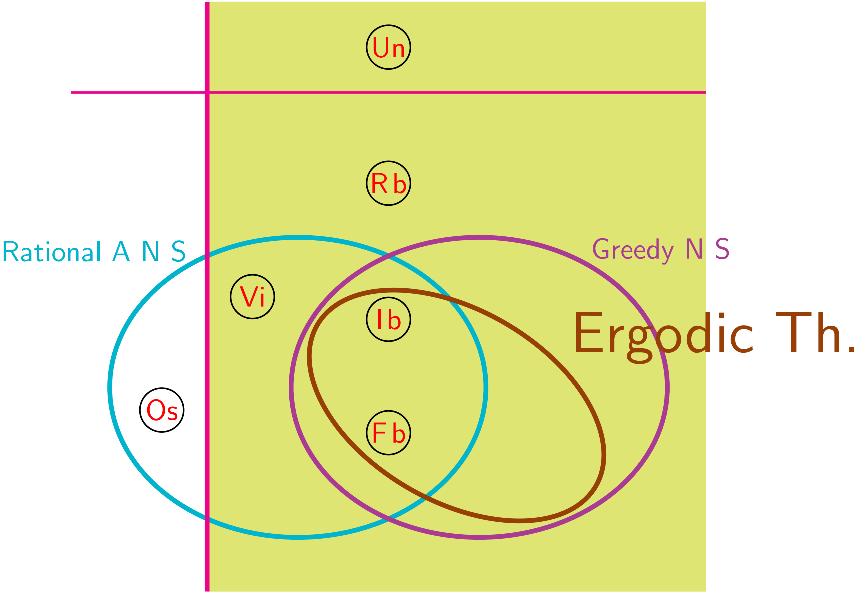

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An unmistakable fit

Our problem

$$\text{Does} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(i) \quad \text{exist ?}$$

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A rewriting

$$\text{Does} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(\mathrm{Succ}_L^i(\varepsilon)) \quad \text{exist ?}$$

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The Ergodic Theorem

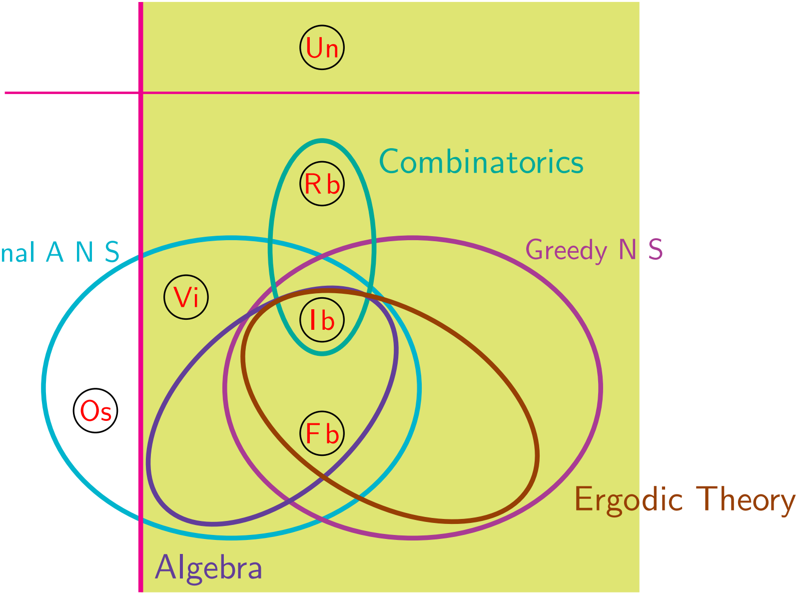
Theorem (Birkhoff 31)

Let $(\mathcal{K}, \tau)$ be a dynamical system, μ a τ-invariant measure on $\mathcal{K}$ and $f : \mathcal{K} \to \mathbb{R}$ in $\mathrm{L}^1(\mu)$ (f is absolutely μ-integrable).
If $(\mathcal{K}, \tau)$ is ergodic, then, for μ-almost all s in $\mathcal{K}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(s)) = \int_{\mathcal{K}} f \, d\mu \quad . \qquad (*)$$

If $(\mathcal{K}, \tau)$ is uniquely ergodic and if f and τ are continuous, then (*) holds for every s in $\mathcal{K}$.

Roadmap



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Comme il y a une infinité de choses sages
qui sont menées de manière très folle,
il y a aussi des folies qui sont menées de manière très sage.

MONTESQUIEU

Just as wise ends are oftentimes sought
in the most foolish way,
so foolishness is sometimes sought with great wisdom.

Translation by REUBEN THOMAS

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  equipped with $\tau\colon \mathcal{K} \to \mathcal{K}$

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  $s = \cdots s_2 s_1 s_0$ and $s_{[\ell,j]} = s_\ell s_{\ell-1} \cdots s_j$

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Turning a numeration system into a dynamical system

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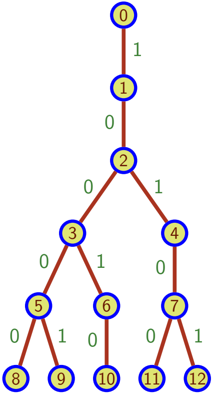

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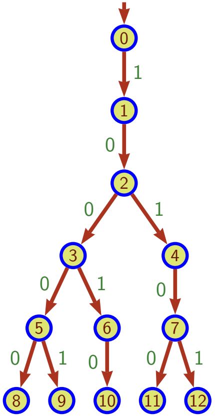

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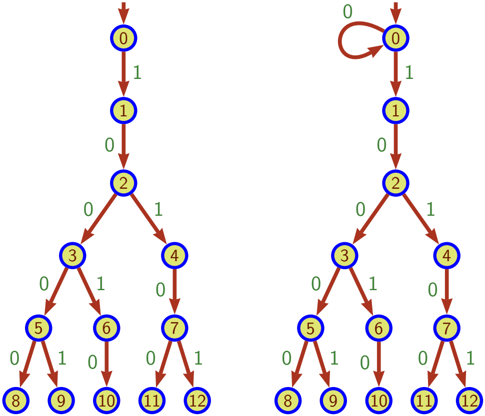

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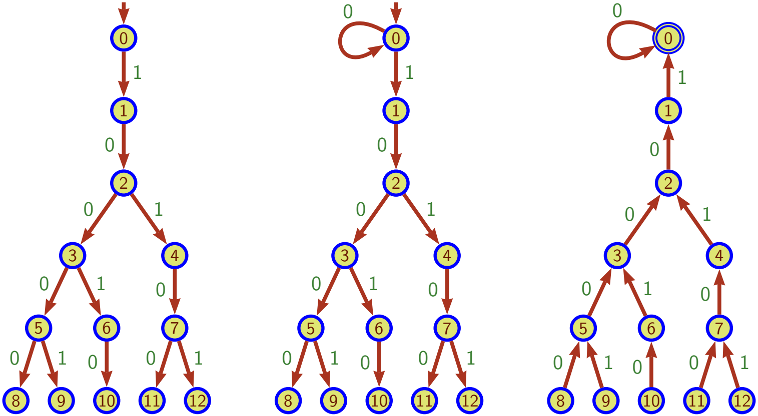

Turning a numeration system into a dynamical system

Definition of the odometer

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$$\forall s \in {}^{\omega}\!A \qquad \mathsf{cp}_L(s) = \Delta(s, \tau_L(s))$$

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Extension of the carry propagation

Proposition

If τ_L *is continuous,*
then cp_L *is continuous at any point where it takes finite values.*

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- $0^* L_G$ is is closed under right factor and
$$\mathcal{K}_G = \overline{{}^\omega 0\, L_G} = \left\{ s \in {}^\omega A \;\middle|\; \forall j \in \mathbb{N} \qquad s_{[j,0]} \in 0^* L_G \right\}$$

Ergodicity of greedy numeration systems

Theorem (Barat–Grabner 16, Grabner–Liardet–Tichy 95)

Let G *be a GNS.*

For every s *in* $\mathcal{K}_G$, $\lim_{j\to\infty} \mathrm{Succ}_G(s_{[j,0]})$ *exists and defines the odometer* $\tau_G\colon \mathcal{K}_G \to \mathcal{K}_G$ *:*

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A GNS G is said to be *exponential* if there exist two real constants $\alpha > 1$ and $C > 0$ such that $G_\ell \sim C\alpha^\ell$ when ℓ tends to infinity.

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Theorem (Barat–Downarowicz–Liardet 02)

If G *is an exponential GNS,*

then the dynamical system $(\mathcal{K}_G, \tau_G)$ *is uniquely ergodic.*

Carry propagation in greedy numeration systems

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If G is an exponential GNS, then CP_G exists.

Carry propagation in greedy numeration systems

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Corollary

If G is an exponential GNS with $G_\ell \sim C\,\alpha^\ell$ and if L_G is PCE*,*
then CP_G exists and $\mathsf{CP}_G = \dfrac{\alpha}{\alpha - 1}$.